

10. Spherical Means and Subharmonic functions.

(a) Show using the definition of integration on a submanifold, that

$$\int_{\partial B(a,r)} f(x) d\sigma(x) = r^{n-1} \int_{\partial B(0,1)} f(a + rz) d\sigma(z) \quad (3 \text{ Points})$$

We introduce the follow general notation for spherical means

$$M(f, a, r) := \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(a,r)} f(x) d\sigma(x),$$

where ω_n is the volume of the unit ball. Show the following properties of the spherical mean.

(b) If c is a constant, $M(c, a, r) = c$. (1 Point)

(c) If $f \leq g$, then $M(f, a, r) \leq M(g, a, r)$, and $|M(f, a, r)| \leq M(|f|, a, r)$. (1 Point)

(d) If f is continuous at a , $\lim_{r \rightarrow 0^+} M(f, a, r) = f(a)$. (2 Points)

Let $\Omega \subset \mathbb{R}^n$ be an open connected domain. A twice continuously differentiable function $v : \bar{\Omega} \rightarrow \mathbb{R}$ is called *subharmonic*, when $-\Delta v \leq 0$ on Ω .

(e) Let $v : \bar{\Omega} \rightarrow \mathbb{R}^n$ be subharmonic. Show for all $x \in \Omega$ and $r > 0$ with $B(x, r) \subset \Omega$ that $v(x) \leq M(v, x, r)$. [Hint: Adapt the proof of the mean value property] (3 Points)

(f) Prove the strong maximum principle for subharmonic functions: If v has a maximum on Ω then v constant. (2 Points)

11. Fundamental solution of the Laplace equation.

Let $n \geq 2$. In this question we investigate a function known as the fundamental solution of the Laplace equation:

$$\Phi : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}, x \mapsto \begin{cases} -\frac{1}{2\pi} \log(\|x\|) & \text{for } n = 2 \\ \frac{1}{n(n-2)\omega_n} \|x\|^{2-n} & \text{for } n \geq 3 \end{cases}$$

(a) Let $u \in C^2(\mathbb{R}^n \setminus \{0\})$ be rotationally symmetric. This means that $u(x) = v(\|x\|)$ for some twice continuously differentiable function $v : [0, \infty) \rightarrow \mathbb{R}$. Show that such solutions of the Laplace equation $\Delta u = 0$ have the form $A\Phi + B$ for constants $A, B \in \mathbb{R}$.

Hint: Use the Laplacian in n -dimensions in spherical coordinates (look it up). (2 Points)

(b) Calculate

$$\nabla \Phi = -\frac{1}{n\omega_n} \frac{x}{\|x\|^n}.$$

(2 Bonus Points)

- (c) The fundamental solution Φ is chosen from this set of solutions to have two properties. The first is that it vanishes at infinity ($B = 0$). The second property ($A = 1$) is that

$$\int_{\partial B(0,r)} \nabla \Phi \cdot N \, d\sigma = -1$$

for all radii. Verify this.

(2 Points)

- (d) Can you see an easy proof that the integral in the previous question is independent of the radius?

(2 Bonus Points)

Why is this function called the fundamental solution? Because in terms of distributions $-\Delta \Phi = \delta$ (using Φ also as the name of the corresponding distribution F_Φ). We will now prove this.

- (e) Check that Φ is locally integrable, so that it does indeed define a distribution. (1 Point)
- (f) For a test function ψ , why does $-\Delta \Phi(\psi) = -\Phi(\Delta \psi)$? (1 Point)
- (g) Separate the integral $-\Phi(\Delta \psi)$ into one part that contains the singularity from Φ and another part that is singularity free:

$$I_\varepsilon := - \int_{B(0,\varepsilon)} \Phi \Delta \psi,$$

$$J_\varepsilon := - \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi \Delta \psi.$$

Show that $\lim_{\varepsilon \rightarrow 0^+} I_\varepsilon = 0$.

(2 Points)

- (h) For every J_ε , prove that the following estimate holds:

$$J_\varepsilon = - \int_{\partial B(0,\varepsilon)} \psi \nabla \Phi \cdot N \, d\sigma + L_\varepsilon,$$

where L_ε is some expression that converges to zero as $\varepsilon \rightarrow 0$.

[Hint: Green's second formula.]

(3 Points)

- (i) Finally, prove $\lim_{\varepsilon \rightarrow 0} J_\varepsilon = \psi(0)$.

(2 Points)

In total we showed that $-\Delta \Phi(\psi) = \psi(0) = \delta(\psi)$ for all test functions ψ . This shows that the negative Laplacian of Φ as a distribution is indeed the delta distribution.

What is the fundamental solution good for? It gives us a solution of the Poisson equation $-\Delta u = f$ in \mathbb{R}^n .

- (j) If F is a distribution with compact support, why does $U = F * \Phi$ solve the Poisson equation $-\Delta U = F$ in the sense of distributions? (1 Point)
- (k) More generally, if L is a linear differential operator, a distribution Φ is called a fundamental solution if $L\Phi = \delta$. Give a solution to the inhomogeneous equation $LU = F$. (1 Bonus Point)