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Partial Differential Equations
Exercise sheet 2

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6. Extension of Continuous Linear Operators. Let X be a normed vector space and $\bar{X}$ its completion. Let Y be a complete normed vector space. Suppose that $L : X \to Y$ is a continuous linear operator. This means that there is a constant C such that $\|Lx\| \leq C\|x\|$ for all $x \in X$. Show that there is a unique continuous linear operator $\bar{L} : \bar{X} \to Y$ extending L (ie $\bar{L}x = Lx$ for all $x \in X$). *(3 Points)*

Solution. Every element of $x \in \bar{X}$ is the limit of a sequence $x_n \in X$. Since we want $\bar{L}$ to be continuous, we are forced to define $\bar{L}x := \lim Lx_n$. We see that Lx_n is a Cauchy sequence

$$\|Lx_n - Lx_m\| = \|L(x_n - x_m)\| \leq C\|x_n - x_m\| \to 0$$

and therefore it converges in Y, so this definition produces a value. If x'_n is another sequence converging to x, then $L(x_n - x'_n)$ is a null sequence, so this other sequence gives the same value. Hence this definition of $\bar{L}$ is well-defined.

It's easy to see it's linear. It's trivial that it's an extension of L. Since the norm is continuous

$$\|\bar{L}x\| = \lim \|Lx_n\| \leq \lim C\|x_n\| = C\|x\|.$$

If there is another continuous extension, then the difference is a continuous linear function that vanishes on a dense subset. This difference therefore must be zero.

7. Distributions I.

(a) Show directly from Definition 2.6 that the Heaviside distribution

$$H : C_0^\infty(\mathbb{R}) \to \mathbb{R},\ \phi \mapsto \int_0^\infty \phi(x)\,\mathrm{d}x$$

is a distribution on $\mathbb{R}$. *(2 Points)*

(b) By the definition of the derivative of a distribution

$$\partial H(\phi) = -H(\partial\phi) = -\int_0^\infty \phi'(x)\,\mathrm{d}x.$$

Simplify this expression in order to give a description of ∂H. (∂ here is the derivative in one-dimension. It seems weird to use an index.) *(3 Points)*

(c) What is the support of ∂H (in the sense of distributions)? Why does this show that there is no function $f \in L^1_{loc}(\mathbb{R})$ with $\partial H = F_f$? *(3 Points)*

(d) Consider a function $f \in C_0^\infty(\mathbb{R}^n)$. Show that $\partial_i(F_f) = F_{\partial_i f}$. What is the connection to Exercise 6?

(2 Points)

Solution.

(a) The integral is always finite because ϕ has compact support. The integral is a linear operator. It only remains to show that H is continuous with respect to the semi-norms. Choose any compact set K and test function ϕ with support in K. Then

$$|H(\phi)| \leq \int_0^\infty |\phi| \leq \int_{\mathbb{R}} |\phi| \leq \mu(K) \|\phi\|_\infty = \mu(K) \|\phi\|_{K,0}$$

(b) If we apply integration by parts we get $-\phi(\infty) + \phi(0)$. Since ϕ has compact support, the first term is zero. Therefore $H(\phi) = \phi(0)$. This is the delta distribution.

(c) By definition, $(\operatorname{supp}\delta)^c$ is the union of all open sets U such that $\delta(\phi) = 0$ for every test function ϕ with support in U. If ϕ is supported in $\mathbb{R} \setminus \{0\}$ then $\delta(\phi) = \phi(0) = 0$. This shows that $\mathbb{R} \setminus \{0\} \subseteq (\operatorname{supp}\delta)^c$. Conversely, if $0 \in U$ then there is an $\varepsilon > 0$ such that the bump function $\phi_{B(0,\varepsilon)}$ has support in U. Because $\phi_{B(0,\varepsilon)}(0) \neq 0$ we conclude that this U is not part of the union that comprises $(\operatorname{supp}\delta)^c$ and thus $0 \notin (\operatorname{supp}\delta)^c$. We have shown that $(\operatorname{supp}\delta)^c = \mathbb{R} \setminus \{0\}$.

$\operatorname{supp}\delta = \{0\}$. Therefore, if $\delta = F_f$ then $\operatorname{supp} f = \{0\}$. But if a function is supported only at one point, then it is equivalent to 0 in the sense of L^1 functions. Since the delta distribution is not the zero distribution, it can not be from a function.

(d)

$$\partial_i(F_f)(\phi) = -F_f(\partial_i\phi) = -\int_{\mathbb{R}^n} f\partial_i\phi = \int_{\mathbb{R}^n} \partial_i f\phi = F_{\partial_i f}(\phi).$$

8. On Convolutions.

(a) Let $f(x) = 1$ for $-1 \leq x \leq 1$ and 0 otherwise. Compute $f * f$. *(2 Points)*

(b) Show that the convolution of C_0^∞-functions on $\mathbb{R}^n$ is a bilinear, commutative, and associative operation. *(1+2 Points + 2 Bonus Points)*

(c) Denote a constant function on $\mathbb{R}$ by 1. The Heaviside function $H : \mathbb{R} \to \mathbb{R}$ is defined as $H(x) := 1$ for $x \geq 0$ and $H(x) := 0$ for $x < 0$. The derivative of the Dirac distribution δ' acts by $\delta'(\phi) = -\phi'(0)$. Let $\phi \in C_0^\infty(\mathbb{R})$ be a test function.

- **(i)** Consider the distribution $\phi * P\delta'$. Which result from the script tells us that this distribution comes from a smooth function, even though δ' does not? *(1 Point)*
- **(ii)** Prove that $\phi * P\delta' = F_{-\phi'}$. *(3 Points)*
- **(iii)** Thereby show that $H * \delta' = \delta$ and $\delta' * 1 = 0$. *(2 Points)*
- **(iv)** Complete the calculation of both $(H * \delta') * 1$ and $H * (\delta' * 1)$ in the sense of distributions and see that they are not equal. This shows that the convolution of distributions with non-compact support (on $\mathbb{R}$) is not necessarily associative, even when it is well-defined. *(1 Point)*

Solution.

(a)

$$f * f(x) = \int_{\mathbb{R}} f(x-y)f(y)\,dy = \int_{-1}^{1} f(x-y)\,dy = \int_{x+1}^{x-1} f(z)\,(-dz) = \int_{x-1}^{x+1} f(z)\,dz$$
$$= \int_{[x-1,x+1]\cap[-1,1]} 1\,dz = \text{len}([x-1,x+1]\cap[-1,1]).$$

(b) Commutativity follows from the substitution $z = x - y$. So it is enough to check bilinearity in the first argument, which is clear. The tricky thing is associativity. The easiest way is to try to reduce both integrals to a common form:

$$(f*g)*h(x) = \int (f*g)(x-z)\,h(z)\,dz = \int \left(\int f(x-z-y)g(y)\,dy\right) h(z)\,dz$$
$$= \iint f(x-y-z)g(y)h(z)\,dy\,dz,$$
$$f*(g*h)(x) = \int f(x-y)\,g*h(y)\,dy = \int f(x-y)\left(\int g(y-z)h(z)\,dz\right)\,dy$$
$$= \iint f(x-y)g(y-z)h(z)\,dz\,dy \quad \text{let } y = w+z, z = z$$
$$= \iint f(x-w-z)g(w)h(z)\,dz\,dw.$$

Ask yourself, where have we used the assumption that the functions are compactly supported?

(c) **(i)** Lemma 1.11: The convolution of a distribution and a smooth function with compact support is a smooth function.

(ii) Let ψ be a test function. By defintion of convolution of distributions $\phi * P\delta'(\psi) = P\delta'(P\phi * \psi)$. We should therefore investigate the argument.

$$P\phi * \psi(x) = \int P\phi(x-y)\psi(y)\,dy = \int \phi(y-x)\psi(y)\,dy.$$

Call this function $j(x)$ for clarity. Now we must act $P\delta'$ on this. We first unwind the definition of P: $P\delta'(j) = \delta'(Pj)$. We see

$$Pj(x) = j(-x) = \int \phi(y+x)\psi(y)\,dy$$
$$\delta'(Pj) = -\left.\frac{d}{dx}\right|_{x=0} \int \phi(y+x)\psi(y)\,dy = \int \Big[-\phi'(y)\Big]\psi(y)\,dy.$$

In other words

$$\phi * P\delta' : \psi \mapsto \int \Big[-\phi'(y)\Big]\psi(y)\,dy.$$

Therefore we see that $\phi * P\delta'$ acts on a test function ψ by integrating it along with $-\phi'$. Hence this distribution comes from $-\phi'$.

(iii) Observe in the definition of the convolution of two distributions $F * G(\phi) = F(\phi * PG)$ that the result of the convolution inside F needs to be interpreted as a test function, not a distribution. Because of Lemma 1.11, this is possible since G has compact support.

$$H * \delta'(\phi) = H(\phi * P\delta') = H(-\phi') = \int_0^\infty -\phi'(y)\, dy$$
$$= \phi(0).$$

For the second convolution, we interpret this as the convolution of a distribution and a smooth function $g * F(\phi) = F(\phi * Pg)$:

$$\delta' * 1(\phi) = \delta'(\phi * P1)$$
$$\phi * P1(x) = \int_{-\infty}^\infty \phi(y) \cdot 1\, dy = \int_{-\infty}^\infty \phi(y)\, dy, \text{ a constant in } x$$
$$\delta' * 1(\phi) = -0 = 0.$$

(iv) $(H * \delta') * 1 = \delta * 1 = 1$ and $H * (\delta' * 1) = H * 0 = 0$.

9. Distributions II.

(a) Show that
$$V(\phi) = \int_0^\infty \frac{\phi(x) - \phi(-x)}{x}\, dx$$
is a distribution on $\mathbb{R}$. Hint: Split the integral into $[0, 1]$ and $[1, \infty]$ and use the mean value theorem. *(2 Bonus Points)*

(b) What is the relation of V to x^{-1}? *(1 Bonus Point)*

(c) Show that the function $u : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \|x\|^k$ is a locally integrable function for $k > -n$. *(2 Bonus Points)*

(d) Let $n = 3$ and $k = -1$. Let $U = F_u$ be the distribution associated to u. It follows from 5(b) that $\partial_i u = -x_i\|x\|^{-3}$, which is also locally integrable, so expect $\partial_i U$ to correspond to $\partial_i u$. However we only know this correspondence holds in situations like Exercise 7(d). Using careful manipulation of the integrals (in particular, cut-out a ball $B(0, \varepsilon)$) show that our expectation holds. *(4 Bonus Points)*

Solution.

(a) V is clearly linear. One can use L'Hopital's rule to see that the integrand is continuous at 0, and therefore integrable. Hence V defines a linear functional, and it remains to show that it is continuous. Let K be a compact set and ϕ a test function on K. We can bound K in the interval $[-R, R]$. Let us therefore split the integral into two parts

$$V(\phi) = \int_0^1 \frac{\phi(x) - \phi(-x)}{x}\, dx + \int_1^R \frac{\phi(x) - \phi(-x)}{x}\, dx.$$

The second integral is easily bound

$$\left|\int_1^R \frac{\phi(x)-\phi(-x)}{x}\,dx\right| \le 2\|\phi\|_{K,0}\int_1^R dx \le 2R\|\phi\|_{K,0}$$

By the mean value theorem, for all x there is a $y \in [-x, x]$ such that

$$\phi'(y) = \frac{\phi(x)-\phi(-x)}{2x}.$$

This allows us to bound the first integral

$$\left|\int_0^1 \frac{\phi(x)-\phi(-x)}{x}\,dx\right| \le 2\|\phi\|_{K,1}$$

Together this gives $|V(\phi)| \le 2R\|\phi\|_{K,0} + 2\|\phi\|_{K,1}$.

(b) x^{-1} is not a local integrable function on $\mathbb{R}$ so does not correspond to a distribution. However, if $0 \notin \operatorname{supp}\phi$ then $V(\phi) = F_{x^{-1}}(\phi)$. So we can think of V as one possible extension of x^{-1}.

(c) Away from the origin u is continuous, so of course it is locally integrable. So we just need to consider an open set containing the origin:

$$\int_{B(0,1)} |u(x)|\,dx = \int_0^1 \int_{B(0,r)} r^k\,d\sigma\,dr = \int_0^1 r^k\,n\omega_n r^{n-1}\,dr = n\omega_n \int_0^1 r^{k+n-1}\,dr$$

is finite if $k > -n$.

(d) By definition

$$\begin{aligned}
\partial_i U(\phi) &= -U(\partial_i\phi) = -\int_{\mathbb{R}^3} u\,\partial_i\phi \\
&= -\int_{B(0,\varepsilon)} u\,\partial_i\phi - \int_{\mathbb{R}^3\setminus B(0,\varepsilon)} u\,\partial_i\phi \\
&= -\int_{B(0,\varepsilon)} u\,\partial_i\phi - \int_{\mathbb{R}^3\setminus B(0,\varepsilon)} \partial_i(u\phi) + \int_{\mathbb{R}^3\setminus B(0,\varepsilon)} \partial_i u\,\phi \\
&= -\int_{B(0,\varepsilon)} u\,\partial_i\phi + \int_{\partial B(0,\varepsilon)} u\phi\,x_i\|x\|^{-1}\,d\sigma + \int_{\mathbb{R}^3\setminus B(0,\varepsilon)} \partial_i u\,\phi
\end{aligned}$$

We examine the three integrals in the limit as $\varepsilon \to 0$. The first integral we know goes to zero, because we can pull out $\|\partial_i\phi\|_\infty$ and then we know that u is integrable. The idea for the second integral is similar

$$\left|\int_{\partial B(0,\varepsilon)} u\phi\,x_i\|x\|^{-1}\,d\sigma\right| \le \|\phi\|_\infty\varepsilon^{-2}\int_{\partial B(0,\varepsilon)} |x_i|\,d\sigma \le \|\phi\|_\infty\varepsilon^{-2} \times \varepsilon\,4\pi\varepsilon^2 \to 0.$$

The third term is exactly what we want. So taking limits

$$\partial_i U(\phi) = 0 + 0 + \lim_{\varepsilon\to 0}\int_{\mathbb{R}^3\setminus B(0,\varepsilon)} \partial_i u\,\phi = \int_{\mathbb{R}^3} \partial_i u\,\phi = F_{\partial_i u}(\phi)$$