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Partial Differential Equations
Exercise sheet 1

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1. Bumpy Road

Optional: Give an example of a function $u : \Omega \subset \mathbb{R} \to \mathbb{R}$ that is

(a) continuous but not differentiable.

(b) differentiable but not continuously differentiable.

(c) belongs to C^k but not C^{k+1}.

Solution.

(a) $u(x) = |x|$.

(b) $u(x) = x^2 \sin x^{-1}$ for $x \neq 0$ and $u(0) = 0$. For $x \neq 0$ it is differentiable with $u'(x) = 2x \sin x^{-1} - \cos x^{-1}$.

$$u'(0) = \lim_{h \to 0} \frac{h^2 \sin h^{-1} - 0}{h - 0} = 0.$$

So the derivative exists everywhere, but is not continuous at zero.

(c) These spaces can be defined inductively: $u \in C^k$ if and only if $u' \in C^{k-1}$ (for $k \geq 1$. The space C^0 is the space of continuous functions. We already have seen that $u_0(x) = |x| \in C^0$ but is not differentiable. If we integrate this, $u_k(x) = \int_0^x u_{k-1}(t)\ \mathrm{d}t$, then we know that $u_k' = u_{k-1} \in C^{k-1} \setminus C^k$ and therefore $u_k \in C^k \setminus C^{k+1}$.

You may recall from Analysis II that there is a somewhat complicated relation between whether a function is partially differentiable and differentiable. However, a function is partially differentiable and all its partial derivatives are continuous if and only if it is continuously differentiable. This is a good reason to consider the space of continuously differentiable functions.

2. Vector Operators

Optional: Write in terms of components the formulas for the gradient ∇, the divergence $\nabla\cdot$, and the Laplacian $\triangle$.

Solution. Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a scalar valued function and $F : \Omega \to \mathbb{R}^n$ a vector valued function.

$$\nabla u = \left(\frac{\partial u}{\partial x_i} \right)_i$$

$$\nabla \cdot F = \sum_i \frac{\partial F_i}{\partial x_i}$$

$$\triangle u = \nabla \cdot (\nabla u) = \sum_i \frac{\partial^2 u}{\partial x_i^2}.$$

3. The linear transport equation

Let $b \in \mathbb{R}^n$. The (homogeneous) linear transport equation with direction b is given by the following partial differential equation of first order:

$$\dot{u} + b \cdot \nabla u = 0 \; . \qquad (*)$$

This is a differential equation of $u = u(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, where $\dot{u}$ denotes the derivative of u with respect to $t \in \mathbb{R}$ and the gradient ∇u is taken with respect to $x \in \mathbb{R}^n$.

(a) Suppose that $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a C^1 solution of $(*)$. Show that u is constant on each of the parallel lines with direction $(b,1) \in \mathbb{R}^n \times \mathbb{R}$. (Hint: Choose a line and parameterise it by s. Use the chain rule.) *(4 points)*

(b) Let $g \in C^1(\mathbb{R}^n)$. Prove that $u(x,t) := g(x - tb)$ is the *unique* solution of $(*)$ satisfying $u(\cdot, 0) = g$. *(5 points)*

Solution.

(a) Consider the line parameterised by $s \mapsto (sb, s)$ and the value of the function u on this line $z(s) := u(sb, s)$. By the chain rule

$$\frac{dz}{ds} = \sum_i \frac{\partial u}{\partial x_i} b_i + \frac{\partial u}{\partial t} 1 = \nabla u \cdot b + \dot{u} = 0.$$

We see that u is constant on each of these lines.

(b) We first see that the given u is a solution. Again by the chain rule

$$\nabla u \cdot b + \dot{u} = \nabla g \cdot b + \nabla g \cdot (0 - b) = 0.$$

To show uniqueness, we first apply a standard trick (this trick is not strictly necessary, but it makes it a little easier to explain the crux of the argument). Suppose that there was another solution v. Because this is a linear PDE, the difference $w = u - v$ is also a solution to the transport equation. Moreover, $w(x,0) = g(x) - g(x) = 0$. Because every point of $\mathbb{R}^n \times \mathbb{R}$ belongs to a line with direction $(b,1)$ and all such lines intersect the plane $t = 0$, by part (a) we conclude that w is zero at every point. Therefore $u = v$.

4. In Colour.

Let Ω be a region in $\mathbb{R}^n$ and N the outer unit normal vector field on $\partial\Omega$. Let u, v be two C^2 real-valued functions on $\overline{\Omega}$.

(a) Show $v \triangle u = \nabla \cdot (v \nabla u) - \nabla u \cdot \nabla v$. *(2 points)*

(b) Prove the first Green formula

$$\int_\Omega v \triangle u \; dx = -\int_\Omega \nabla u \cdot \nabla v \; dx + \int_{\partial\Omega} v \nabla u \cdot N \; d\sigma.$$

(2 points)

(c) Using the first Green formula, prove the second Green formula

$$\int_\Omega (v\triangle u - u\triangle v)\, dx = \int_{\partial\Omega} (v\nabla u - u\nabla v)\cdot N\, d\sigma.$$

(1 points)

(d) Suppose further that v has support in Ω. This means that $\overline{\{x\in\Omega \mid v(x)\neq 0\}} \subsetneq \Omega$. Prove that

$$\int_\Omega v\triangle u\, dx = \int_\Omega u\triangle v\, dx$$

(1 points)

Solution.

(a) If you are not familiar with vector calculus, remember that you can also write the expression out using summations signs and then use normal calculus rules. Indeed, this is where the vector calculus rules come from.

Start with $g\nabla f$. Its ith component is $g\partial_i f$. Therefore

$$\nabla\cdot(g\nabla f) = \sum_i \partial_i(g\partial_i f) = \sum_i \partial_i g\partial_i f + g\partial_i^2 f = \nabla g\cdot\nabla f + g\triangle f$$

(b) Integrate the expression from the previous part and apply the divergence theorem

$$\int_\Omega \nabla\cdot(v\nabla u)\, dx = \int_{\partial\Omega} v\nabla u\cdot N\, d\sigma.$$

(c) The second Greens formula is simply a symmetrised version of the first:

$$\begin{aligned}\int_\Omega (v\triangle u - u\triangle v)\, dx &= -\int_\Omega (\nabla u\cdot\nabla v - \nabla v\cdot\nabla u)\, dx + \int_{\partial\Omega}(v\nabla u - u\nabla v)\cdot N\, d\sigma\\ &= \int_{\partial\Omega}(v\nabla u - u\nabla v)\cdot N\, d\sigma.\end{aligned}$$

(d) Since v has compact support, it and its derivatives must vanish on $\partial\Omega$. Therefore the right hand side of (c) is zero. The result follows.

5. Laplacian and Laplace equation Laplace's equation is $\triangle u = 0$. A solution to Laplace's equation is called a harmonic function. We will discuss harmonic functions in further detail in the next chapter.

(a) Let $u, v : \Omega \to \mathbb{R}$ be harmonic functions. Show that the function $w(x) := u(x)v(x)$ is harmonic exactly when $\nabla u \perp \nabla v$. *(2 points)*

(b) Consider the function $u : \mathbb{R}^n \to \mathbb{R}, x \mapsto \|x\|$. Compute its gradient and Laplacian. *(3 points)*

(c) Optional: Let $u : \mathbb{R}^2 \to \mathbb{R}$ be twice-differentiable. Show for polar coordinates $x = r\cos(\varphi)$, $y = r\sin(\varphi)$ that

$$\triangle u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}.$$

(d) Optional: Let $u : \mathbb{R}^3 \to \mathbb{R}$ be twice differentiable.

(i) Show for cylindrical coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$ that

$$\triangle u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$

(ii) Show for spherical coordinates $x = r\sin(\theta)\cos(\varphi)$, $y = r\sin(\theta)\sin(\varphi)$, $z = r\cos(\theta)$ that

$$\triangle u = \frac{1}{r^2\sin(\theta)}\left[\frac{\partial}{\partial r}\left(r^2\sin(\theta)\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial u}{\partial \theta}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{\sin(\theta)}\frac{\partial u}{\partial \varphi}\right)\right].$$

(e) Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain. Let $u \in C^2(\overline{\Omega})$ be a solution of the *boundary value problem*

$$\triangle u = 0 \quad \text{with} \quad u|_{\partial\Omega} = 0\,.$$

Show $u \equiv 0$. *(5 points)*

[Hint: Investigate $\int_\Omega u(\triangle u)\,\mathrm{d}x$ with the help of Green's first formula.]

Solution.

(a) We have $\partial_j^2 w = \partial_j^2 u v + 2\partial_j u \partial_j v + u\partial_j^2 v$. Summing over j gives the relation $\triangle w = \triangle u v + 2\nabla u \cdot \nabla v + u\triangle v$. Since u and v are harmonic, $\triangle w = 2\nabla u \cdot \nabla v$. Hence w is harmonic if and only if ∇u and ∇v are perpendicular (or one is zero, if you don't want to say that the zero vector is perpendicular to all vectors).

(b)

$$\partial_i\|x\| = \partial_i\left(\sum_j x_j^2\right)^{1/2} = \frac{1}{2}\left(\sum_j x_j^2\right)^{-1/2} 2x_i = \frac{x_i}{\|x\|},$$

$$\partial_i^2\|x\| = \partial_i\frac{x_i}{\|x\|} = \frac{\|x\| - x_i^2\|x\|^{-1}}{\|x\|^2} = \frac{\|x\|^2 - x_i^2}{\|x\|^3}$$

It follows that $\nabla\|x\| = \|x\|^{-1}x$ and $\triangle\|x\| = (n-1)\|x\|^{-1}$.

(c) This follows from the chain rule.

(d) These both follow from the chain rule.

(e) We begin by calculating from the hint

$$\begin{aligned} 0 = \int_\Omega u(\triangle u)\,\mathrm{d}x &= \int_\Omega \nabla(u\nabla u) - \nabla u \cdot \nabla u\,\mathrm{d}x \\ &= \int_{\partial\Omega} u\nabla u \cdot N\,\mathrm{d}\sigma - \int_\Omega \|\nabla u\|^2\,\mathrm{d}x \\ &= 0 - \int_\Omega \|\nabla u\|^2\,\mathrm{d}x. \end{aligned}$$

We want to conclude from this that $\|\nabla u(x)\| = 0$ for all points $x \in \Omega$. Suppose there was a point x_0 where it was not zero. By continuity there is neighbourhood U where $\|\nabla u\| > C$ for some positive constant $C < \|\nabla u(x_0)\|$. But then

$$0 = \int_\Omega \|\nabla u\|^2 \geq \int_U \|\nabla u\|^2 \geq C^2 \mu(U) > 0.$$

Since $\nabla u \equiv 0$ then u must be constant. But since it is zero at the boundary, it must be zero everywhere.

The argument in the middle can also be proved with the fundamental lemma of the calculus of variations (Lemma 1.13). This lemma belongs to the section on distributions, so feel free to look at this proof again next week, when we have covered that material. Suppose that $f \geq 0$ and $\int_\Omega f = 0$. Let ϕ be any test function. We can write it in terms of its positive and negative parts $\phi = \phi^+ - \phi^-$ with $\phi^+ := \max\{\phi, 0\}$ and $\phi^- := \max\{-\phi, 0\}$. Then

$$0 \leq \int_\Omega f \phi^+ \leq \sup \phi^+ \int_\Omega f = 0$$

and likewise for ϕ^-. Together this shows that $F_f(\phi) = 0$ and F_f is the zero distribution. But the association of $f \mapsto F_f$ is injective, therefore f must be the zero function.