Doubly Periodic Solutions of the Sinh-Gordon Equation

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Seminar Report

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1 Introduction

Definition 1.1. *The elliptic* ***sinh-Gordon equation*** *is given by*

$$\Delta u + 2\sinh(2u) = 0,$$

where Δ *is the Laplacian of* $\mathbb{R}^2$ *with respect to the Euclidean metric and* $u : \mathbb{R}^2 \to \mathbb{R}$ *is a twice partially differentiable real function.*

We investigate the finite-type class of solutions of this equation, which is parametrized in the following way: for each non-negative integer $g \in \mathbb{N}_0$ - the so-called **spectral genus** - there exists a family of solutions whose complexity increases with g. We investigate solutions of spectral genus $g = 2$.

We first introduce the set of potentials which parametrize the solutions of the sinh-Gordon equation. On this space we define vector fields which induce two commuting flows. The orbits of these flows are called Polynomial Killing fields. The potential's determinant is an integral of motion along the trajectories of these flows and we study the structure of the so-called isospectral sets in dependence on the position of determinant's roots. Lastly, we investigate the dependence of the period lattice on the isospectral sets.

This seminar report is based on the paper *Solutions of the Sinh-Gordon Equation of Spectral Genus Two and Constrained Willmore Tori I* by M. Knopf, R. Peña Hoepner and M.U. Schmidt [2] and uses numerous details from the Master's Thesis by R. Peña Hoepner [1].

2 Potentials and Polynomial Killing Fields

Definition 2.1. *The **set of potentials** is the set of cubic polynomials with matrix-valued coefficients:*

$$\mathcal{P}_2 := \left\{ \zeta = \begin{pmatrix} \alpha\lambda - \overline{\alpha}\lambda^2 & -\gamma^{-1} + \beta\lambda - \gamma\lambda^2 \\ \gamma\lambda - \overline{\beta}\lambda^2 + \gamma^{-1}\lambda^3 & -\alpha\lambda + \overline{\alpha}\lambda^2 \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{C}, \gamma \in \mathbb{R}^+ \right\}.$$

Every $\zeta_\lambda \in \mathcal{P}_2$ satisfies the **reality condition**:

$$\lambda^3 \overline{\zeta}^t_{1/\overline{\lambda}} = \lambda^3 \begin{pmatrix} \overline{\alpha}\lambda^{-1} - \alpha\lambda^{-2} & \gamma\lambda^{-1} - \beta\lambda^{-2} + \gamma^{-1}\lambda^{-3} \\ -\gamma^{-1} + \overline{\beta}\lambda^{-1} - \gamma\lambda^{-2} & -\overline{\alpha}\lambda^{-1} + \alpha\lambda^{-2} \end{pmatrix} = -\zeta_\lambda.$$

We define the polynomials $a \in \mathbb{C}^4[\lambda]$ of fourth degree as

$$\begin{aligned} \det \zeta &= (\alpha\lambda - \overline{\alpha}\lambda^2)(\overline{\alpha}\lambda^2 - \alpha\lambda) + (\gamma\lambda - \overline{\beta}\lambda^2 + \gamma^{-1}\lambda^3)(\gamma^{-1} - \beta\lambda + \gamma\lambda^2) \\ &= \lambda(\lambda^4 + (-\overline{\alpha}^2 - \overline{\beta}\gamma - \beta\gamma^{-1})\lambda^3 + (2\alpha\overline{\alpha} + \gamma^2 + \beta\overline{\beta} + \gamma^{-2})\lambda^2 + (-\alpha^2 - \beta\gamma - \overline{\beta}\gamma^{-1})\lambda + 1) \\ &=: \lambda a(\lambda). \end{aligned}$$

We can then write $a(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + \overline{a}_1\lambda + 1$ with

$$a_1 = -\overline{\alpha}^2 - \beta\gamma - \beta\gamma^{-1} \in \mathbb{C},\ a_2 = 2\alpha\overline{\alpha} + \gamma^2 + \beta\overline{\beta} + \gamma^{-2} \in \mathbb{R}^+.$$

Theorem 2.2. *The following sets are the same:*

$$\begin{aligned} \mathcal{M}_2 : &= \left\{ a \in \mathbb{C}^4[\lambda] \mid \lambda a(\lambda) = \det(\zeta_\lambda) \ for\ a\ \zeta_\lambda \in \mathcal{P}_2 \right\} \\ &= \left\{ a \in \mathbb{C}^4 \mid a(0) = 1, \lambda^4 \overline{a(\overline{\lambda}^{-1})} = a(\lambda), \lambda^{-2} a(\lambda) \geq 0\ for\ \lambda \in \mathbb{S}^1 \right\}. \end{aligned}$$

For detailed proof, see [1]

Theorem 2.3. *Let $\zeta_\lambda \in \mathcal{P}_2$ and $\det \zeta_\lambda = \lambda a(\lambda)$ with $a(\lambda) \in \mathcal{M}_2$. If $\tilde{\lambda} \in \mathbb{C}$ is a root of ζ_λ, then $\tilde{\lambda}$ is a double root of $a(\lambda)$. Conversely, if $\tilde{\lambda} \in \mathbb{S}^1$ is a root of $a(\lambda)$, then $\tilde{\lambda}$ is a root of ζ_λ.*

For proof, see [1].

Definition 2.4. ***Polynomial Killing fields*** *are maps $\zeta_\lambda : \mathbb{R}^2 \to \mathcal{P}_2,\ (x, y) \mapsto \zeta_\lambda(x, y)$, which solve the **Lax equations***

$$\frac{\partial \zeta_\lambda}{\partial x} = [\zeta_\lambda, U(\zeta_\lambda)],\ \frac{\partial \zeta_\lambda}{\partial y} = [\zeta_\lambda, V(\zeta_\lambda)]$$

with $\zeta_\lambda(0) = \zeta^0_\lambda \in \mathcal{P}_2$ and

$$U(\zeta_\lambda) = \begin{pmatrix} \frac{\alpha - \overline{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} - \gamma \\ \gamma + \gamma^{-1}\lambda & \frac{\overline{\alpha} - \alpha}{2} \end{pmatrix},$$

$$V(\zeta_\lambda) = i \begin{pmatrix} \frac{\alpha + \overline{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} + \gamma \\ \gamma - \gamma^{-1}\lambda & -\frac{\overline{\alpha} + \alpha}{2} \end{pmatrix}.$$

A polynomial Killing field induces a triple of complex functions

$$\begin{aligned}
\alpha &: \mathbb{R}^2 \to \mathbb{C}, (x,y) \mapsto (\zeta_\lambda(x,y))_\alpha \\
\beta &: \mathbb{R}^2 \to \mathbb{C}, (x,y) \mapsto (\zeta_\lambda(x,y))_\beta \\
\gamma &: \mathbb{R}^2 \to \mathbb{R}^+, (x,y) \mapsto (\zeta_\lambda(x,y))_\gamma,
\end{aligned}$$

which satisfy the following ODE systems

$$\begin{aligned}
\frac{\partial\alpha}{\partial x} &= \gamma^2 + \beta\gamma - \overline{\beta}\gamma^{-1} - \gamma^{-2}, & \frac{\partial\alpha}{\partial y} &= i(\gamma^{-2} + \beta\gamma - \overline{\beta}\gamma^{-1} - \gamma^2) \\
\frac{\partial\beta}{\partial x} &= -\alpha\beta + \overline{\alpha}\beta - 2\alpha\gamma + 2\overline{\alpha}\gamma^{-1}, & \frac{\partial\beta}{\partial y} &= i(-\alpha\beta - \overline{\alpha}\beta + 2\alpha\gamma + 2\overline{\alpha}\gamma^{-1}) \\
\frac{\partial\gamma}{\partial x} &= -\alpha\gamma - \overline{\alpha}\gamma, & \frac{\partial\gamma}{\partial y} &= i(\overline{\alpha}\gamma - \alpha\gamma),
\end{aligned}$$

which we call **modified Lax equations**. They are justified by directly computing the commutators in the Lax equations and comparing their entries with those from ζ_λ. See [1] for more details.

Now we have to answer the question whether such a Polynomial Killing field exists.

Lemma 2.5. *The local flows $\phi_E(x,\zeta_\lambda)$ and $\phi_F(y,\zeta_\lambda)$, induced by the vector fields $E(\zeta_\lambda) := [\zeta_\lambda, U(\zeta_\lambda]$ and $F(\zeta_\lambda) := [\zeta_\lambda, V(\zeta_\lambda)]$ respectively, commute.*

Proof. We show that $[E,F] = 0$ holds. This can be expressed as

$$[E,F](\zeta_\lambda) = \left[\zeta_\lambda, [V(\zeta_\lambda, U(\zeta_\lambda)] + \frac{\partial U(\zeta_\lambda)}{\partial y} - \frac{\partial V(\zeta_\lambda)}{\partial x}\right].$$

Consequently, the idea is to show that $[V(\zeta_\lambda, U(\zeta_\lambda)] + \frac{\partial U(\zeta_\lambda)}{\partial y} - \frac{\partial V(\zeta_\lambda)}{\partial x} = 0$ holds. For more details, see [1]. □

As a result we have shown the following:

$$[U(\zeta_\lambda), V(\zeta_\lambda)] + \frac{\partial V(\zeta_\lambda)}{\partial x} - \frac{\partial U(\zeta_\lambda)}{dy} = 0, \; \frac{\partial^2\zeta_\lambda}{\partial x\partial y} = \frac{\partial^2\zeta_\lambda}{\partial y\partial x}.$$

The first equation is called **Maurer-Cartan equation** and helps understand the link of potentials and polynomial Killing fields to the sinh-Gordon equation:

Theorem 2.6. *The function $u := \ln\gamma$ solves the sinh-Gordon equation.*

Proof. We introduce a new coordinate $z := x + iy$ and express x, y in terms of z:

$$x = \frac{1}{2}(z + \overline{z}), \; y = -\frac{i}{2}(z - \overline{z}).$$

We can then write the complex derivatives with respect to z, $\overline{z}$ as follows:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \; \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

We now define $u := \ln\gamma$, which is equivalent to $\gamma = e^u$. We can now calculate the derivatives with respect to x, y and obtain the derivatives with respect to $z, \overline{z}$:

$$\begin{aligned}
&\frac{\partial u}{\partial x} = \frac{1}{\gamma}\frac{\partial \gamma}{\partial x} = -(\alpha + \overline{\alpha}),\ \frac{\partial u}{\partial y} = \frac{1}{\gamma}\frac{\partial \gamma}{\partial y} = i(\overline{\alpha} - \alpha) \\
&\Rightarrow \frac{\partial u}{\partial z} = -\alpha,\ \frac{\partial u}{\partial \overline{z}} = -\overline{\alpha}.
\end{aligned}$$

Our goal is to express the Maurer-Cartan equation in terms of u and derivatives of u with respect to $z, \overline{z}$, which we denote $u_z, u_{\overline{z}}$:

$$\begin{aligned}
U(\zeta_\lambda) &= \begin{pmatrix} \frac{\alpha - \overline{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} - \gamma \\ \gamma + \gamma^{-1}\lambda & \frac{\overline{\alpha} - \alpha}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(u_{\overline{z}} - u_z) & -e^{-u}\lambda^{-1} - e^u \\ e^u + e^{-u}\lambda & \frac{1}{2}(u_z - u_{\overline{z}}) \end{pmatrix} \\
V(\zeta_\lambda) &= i\begin{pmatrix} \frac{\alpha + \overline{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} + \gamma \\ \gamma - \gamma^{-1}\lambda & -\frac{\overline{\alpha} + \alpha}{2} \end{pmatrix} = i\begin{pmatrix} -\frac{1}{2}(u_z + u_{\overline{z}}) & -e^u\lambda^{-1} + e^u \\ e^u - e^{-u}\lambda & \frac{1}{2}(u_z + u_{\overline{z}}) \end{pmatrix} \\
\frac{\partial U}{\partial y}(\zeta_\lambda) &= -i\begin{pmatrix} \frac{1}{2}(u_{\overline{z}\overline{z}} - 2u_{z\overline{z}} + u_{zz}) & e^{-u}(u_{\overline{z}} - u_z)\lambda^{-1} - e^u(u_{\overline{z}} - u_z) \\ e^u(u_{\overline{z}} - u_z) - e^{-u}(u_{\overline{z}} - u_z)\lambda & \frac{1}{2}(-u_{zz} + 2u_{z\overline{z}} - u_{\overline{z}\overline{z}}) \end{pmatrix} \\
\frac{\partial V}{\partial x}(\zeta_\lambda) &= i\begin{pmatrix} -\frac{1}{2}(u_{\overline{z}\overline{z}} + 2u_{z\overline{z}} + u_{zz}) & e^{-u}(u_{\overline{z}} + u_z)\lambda^{-1} + e^u(u_{\overline{z}} + u_z) \\ e^u(u_{\overline{z}} + u_z) - e^{-u}(u_{\overline{z}} + u_z)\lambda & \frac{1}{2}(u_{zz} + 2u_{z\overline{z}} + u_{\overline{z}\overline{z}}) \end{pmatrix} \\
\frac{\partial V(\zeta_\lambda)}{\partial x} - \frac{\partial U(\zeta_\lambda)}{\partial y} &= i\begin{pmatrix} -2u_{z\overline{z}} & 2\lambda^{-1}e^{-u}u_{\overline{z}} + 2e^u u_z \\ 2e^u u_{\overline{z}} + 2\lambda e^{-u}u_z & 2u_{z\overline{z}} \end{pmatrix} \\
[V(\zeta_\lambda), U(\zeta_\lambda)] &= i\begin{pmatrix} 2(e^{2u} - e^{-2u}) & 2e^{-u}u_{\overline{z}}\lambda^{-1} + 2e^u u_z \\ 2e^u u_{\overline{z}} + 2e^{-u}u_z\lambda & 2(e^{-2u} - e^{2u}) \end{pmatrix}
\end{aligned}$$

We see directly that the Maurer-Cartan equation is satisfied if and only if $u_{z\overline{z}} = e^{-2u} - e^{2u}$ holds. Now we want to see that we can rewrite this equation as the sinh-Gordon equation, so we compute the Laplacian with respect to $z, \overline{z}$:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{zz} + 2u_{z\overline{z}} + u_{\overline{z}\overline{z}} - u_{\overline{z}\overline{z}} + 2u_{z\overline{z}} - u_{zz} = 4u_{z\overline{z}}.$$

If we choose $z' = \frac{1}{2}z$ and conduct the same computations, the Maurer-Cartan equation turns into the sinh-Gordon equation:

$$\Delta u + \sinh(2u) = 0.$$

□

Define $f : \mathcal{P}_2 \to \mathcal{M}_2,\ \zeta_\lambda \mapsto a$, where $\det\zeta_\lambda = \lambda a(\lambda)$. We can rewrite this function equivalently as $f : \mathbb{C} \times \mathbb{C} \times \mathbb{R}^+ \to \mathbb{C} \times \mathbb{R}^+, (\alpha, \beta, \gamma) \mapsto (a_1, a_2)$.

Definition 2.7. *The* ***isospectral sets*** *are level sets of the function f and can be written in the following way:*

$$I(a) = \{\zeta \in \mathcal{P}_2 \mid \det\zeta = \lambda a(\lambda)\}.$$

Theorem 2.8. *For a fixed $a \in \mathcal{M}_2$ the set $I(a)$ is compact.*

Proof. We use Heine-Borel and show closedness and compactness. We define f_1 and f_2 using the projection maps:

$$\begin{aligned}
&\pi_1 : \mathbb{C} \times \mathbb{R}^+ \to \mathbb{C}, (a_1, a_2) \mapsto a_1, \\
&\pi_1 : \mathbb{C} \times \mathbb{R}^+ \to \mathbb{R}^+, (a_1, a_2) \mapsto a_2, \\
&f_1 := \pi_1 \circ f,\ (\alpha, \beta, \gamma) \mapsto a_1 = -\overline{\alpha}^2 - \beta\gamma^{-1} - \overline{\beta}\gamma, \\
&f_2 := \pi_2 \circ f,\ (\alpha, \beta, \gamma) \mapsto a_2 = 2\alpha\overline{\alpha} + \beta\overline{\beta} + \gamma^2 + \gamma^{-2}.
\end{aligned}$$

These functions are continuous, so the pre-images $f_1^{-1}[\{a_1\}]$ and $f_2^{-1}[\{a_2\}]$ are closed, hence the set $I(a)$ written as

$$I(a) = f_1^{-1}[\{a_1\}] \cap f_2^{-1}[\{a_2\}]$$

is closed. For boundedness, we show that $f_2^{-1}[\{a_2\}]$ is bounded. We see directly that

$$f_2^{-1}[\{a_2\}] \subset B\,(0, \sqrt{a_2}) \times B\,(0, \sqrt{a_2}) \times (0, \sqrt{a_2})$$

holds true. This proves the claim. □

Theorem 2.9. *The determinant polynomial $a(\lambda)$ is an **integral of motion** with respect to the Lax equation, meaning it is a constant quantity along the trajectories of the Lax equations.*

Proof. This claim is easily proved by computing the derivatives of a_1, a_2 with respect to x, y, which turn out to be zero. For detailed computations, see [1]. □

Theorem 2.10. *Given any initial value $(\alpha_0, \beta_0, \gamma_0) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^+$ the solutions of the modified Lax equations are global, i.e. well-defined for all $(x, y) \in \mathbb{R}^2$, and bounded. Therefore, given any $\zeta_\lambda \in \mathcal{P}_2$ we obtain a continuous, commutative group action*

$$\phi : \mathbb{R}^2 \ni (x, y) \mapsto \phi(x, y),\ \phi(x, y) : \mathcal{P}_2 \to \mathcal{P}_2,\ \zeta_\lambda \mapsto \phi_F(y, \phi_E(x, \zeta_\lambda)),$$

where $\phi_E(x, \zeta_\lambda)$ and $\phi_F(y, \zeta_\lambda)$ are local flows induced by the vector fields $E(\zeta_\lambda) := [\zeta_\lambda, U(\zeta_\lambda]$ and $F(\zeta_\lambda) := [\zeta_\lambda, V(\zeta_\lambda)]$ respectively.

Proof. For every initial value $(\alpha_0, \beta_0, \gamma_0) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^+$ we can solve the modified Lax equations using classic ODE theory. Using Theorems 2.8 and 2.9 we see that any solution's orbit is contained in a compact set $I(a_0)$, where $a_0(\lambda)$ is the polynomial corresponding to the initial value. So the solutions are defined globally and are bounded.
The following calculations show that the map ϕ defines a group action:

$$\begin{aligned}
&\phi(0,0)(\zeta_\lambda) = \phi_F(0, \phi_E(0, \zeta_\lambda)) = \phi_F(0, \zeta_\lambda) = \zeta_\lambda. \\
&\phi(x_2, y_2)(\phi(x_1, y_1)(\zeta_\lambda)) = \phi_F(y_2, \phi_E(x_2, \phi(x_1, y_1)(\zeta_\lambda)) \\
&= \phi_F(y_2, \phi_E(x_2, \phi_F(y_1, \phi_E(x_1, \zeta_\lambda)) = \phi_F(y_2, \phi_F(y_1, \phi_E(x_2, \phi_E(x_1, \zeta_\lambda)) \\
&= \phi_F(y_2 + y_1, \phi_E(x_2 + x_1, \zeta_\lambda)) = \phi(x_1 + x_2, y_1 + y_2)(\zeta_\lambda).
\end{aligned}$$

Commutativity of the group action ϕ follows directly from commutativity of the flows ϕ_E and ϕ_F. We rewrite the map ϕ as combination of continuous maps, which finally shows continuity:

$$(x, y, \zeta_\lambda) \mapsto (y, x, \zeta_\lambda) \xmapsto{id \times \phi_E} (y, \phi_E(x, \zeta_\lambda)) \xmapsto{\phi_F} \phi_F(y, \phi_E(x, \zeta_\lambda)) = \phi(x, y)(\zeta_\lambda).$$

□

3 The Isospectral Sets

We extend our analysis of the isospectral sets with respect to structural features.
By the Fundamental Theorem of Algebra, $a(\lambda)$ has four (possibly multiple) roots in $\mathbb{C} \setminus \{0\}$.

Theorem 3.1. *$\mathcal{M}_2$ is the disjoint union of the following sets:*

$$\mathcal{M}_2^1 := \{a \in \mathcal{M}_2 \mid a \textit{ has four pairwise distinct simple roots absent } \mathbb{S}^1\},$$
$$\mathcal{M}_2^2 := \{a \in \mathcal{M}_2 \mid a \textit{ has one double roots on } \mathbb{S}^1 \textit{ and two simple roots absent } \mathbb{S}^1\},$$
$$\mathcal{M}_2^3 := \{a \in \mathcal{M}_2 \mid a \textit{ has two distinct double roots on } \mathbb{S}^1\},$$
$$\mathcal{M}_2^4 := \{a \in \mathcal{M}_2 \mid a \textit{ has a fourth-order root on } \mathbb{S}^1\},$$
$$\mathcal{M}_2^5 := \{a \in \mathcal{M}_2 \mid a \textit{ has two distinct double roots absent } \mathbb{S}^1\}.$$

Proof.

1. We first consider the case that $a(\lambda)$ has four pairwise distinct roots. If one of the roots λ_1 were on $\mathbb{S}^1$, then by Theorem 2.3 λ_1 would be a double root of $a(\lambda)$. Hence, all roots are absent $\mathbb{S}^1$ and we are in the case of $\mathcal{M}_2^1$.

2. If $a(\lambda)$ has a double root λ_1 absent $\mathbb{S}^1$, then by reality condition $\lambda^4\overline{a(\bar{\lambda}^{-1})} = a(\lambda)$, $\bar{\lambda}_1^{-1} \neq \lambda_1$ is also a double root absent $\mathbb{S}^1$ and we are in the case of $\mathcal{M}_2^5$.

3. Now we consider the case that $a(\lambda)$ has a double root on $\mathbb{S}^1$. If one of the other roots λ_1 is absent $\mathbb{S}^1$, then again $\bar{\lambda}_1^{-1}$ is the remaining root absent $\mathbb{S}^1$, which is case $\mathcal{M}_2^2$. If on the other hand one of the other roots happens to be on $\mathbb{S}^1$, then it must be a double root. Depending on whether these two double roots coincide, we are either in the case $\mathcal{M}_2^3$ or $\mathcal{M}_2^4$.

□

Theorem 3.2. *(i) For $a(\lambda) \in \mathcal{M}_2^1$ the isospectral sets $I(a)$ are two-dimensional compact submanifolds of $\mathcal{P}_2$. The maps $\phi(x,y)(\zeta_\lambda)$ for given $\zeta_\lambda \in I(a)$ define a transitive group action on the isospectral sets, i.e.*

$$I(a) = \{\phi(x,y)(\zeta_\lambda) \mid (x,y) \in \mathbb{R}^2\}.$$

(ii) For $a(\lambda) \in \mathcal{M}_2^2$ the isospectral sets $I(a)$ are one-dimensional compact subsets of $\mathcal{P}_2$. The maps $\phi(x,y)(\zeta_\lambda)$ for given $\zeta_\lambda \in I(a)$ define a transitive group action on the isospectral sets.

(iii) For $a(\lambda) \in \mathcal{M}_2^3 \cup \mathcal{M}_2^4$ the isospectral set $I(a)$ contains a unique fixed point of the group action. The maps $\phi(x,y)(\zeta_\lambda)$ for given $\zeta_\lambda \in I(a)$ remain constant, i.e. they act transitively in a trivial way.

(iv) For $a(\lambda) \in \mathcal{M}_2^5$ with double roots $\lambda_1, \bar{\lambda}_1^{-1}$ absent $\mathbb{S}^1$ the isospectral set $I(a)$ decomposes into two distinct subsets:

$$I(a) = \{\zeta_\lambda \in I(a) \mid \zeta_{\lambda_1} \neq 0\} \cup \{\zeta_\lambda \in I(a) \mid \zeta_{\lambda_1} = \zeta_{\bar{\lambda}_1^{-1}} = 0\} =: K_a \cup L_a.$$

Here K_a is a two-dimensional non-compact submanifold of $\mathcal{P}_2$ with closure $\overline{K}_a = I(a)$ and L_a contains a single point. On both parts the maps $\phi(x,y)(\zeta_\lambda)$ act transitively for given $\zeta_\lambda \in I(a)$.

The proof can be found in [1] or [2].

Remark: In [2] the proof of Theorem 3.2 (ii) uses the claim that the following map transforms potentials in $\mathcal{P}_1$ to potentials in $\mathcal{P}_2$:

$$\zeta_\lambda = e^{3i\phi} \begin{pmatrix} 1 & 0 \\ 0 & ie^{i\phi} \end{pmatrix} \hat{\zeta}_{\hat{\lambda}} \begin{pmatrix} 1 & 0 \\ 0 & -ie^{-i\phi} \end{pmatrix} (\lambda - e^{2i\phi}), \tag{3.1}$$

where

$$\mathcal{P}_1 := \left\{ \hat{\zeta}_{\hat{\lambda}} = \begin{pmatrix} i\hat{\alpha}\hat{\lambda} & -\hat{\beta}^{-1} - \hat{\beta}\hat{\lambda} \\ \hat{\beta}\hat{\lambda} + \hat{\beta}^{-1}\hat{\lambda}^2 & -i\hat{\alpha}\hat{\lambda} \end{pmatrix} \middle| \ \hat{\alpha} \in \mathbb{R}, \hat{\beta} \in \mathbb{R}^+ \right\}, \ \hat{\lambda} = -e^{2i\phi}\lambda.$$

Direct computation of the right-hand side of 3.1 yields

$$\begin{aligned}
&e^{3i\phi} \begin{pmatrix} 1 & 0 \\ 0 & ie^{i\phi} \end{pmatrix} \hat{\zeta}_{\hat{\lambda}} \begin{pmatrix} 1 & 0 \\ 0 & -ie^{-i\phi} \end{pmatrix} (\lambda - e^{2i\phi}) \\
&= e^{3i\phi}(\lambda - e^{2i\phi}) \begin{pmatrix} 1 & 0 \\ 0 & ie^{i\phi} \end{pmatrix} \begin{pmatrix} i\hat{\alpha}\hat{\lambda} & -\hat{\beta}^{-1} - \hat{\beta}\hat{\lambda} \\ \hat{\beta}\hat{\lambda} + \hat{\beta}^{-1}\hat{\lambda}^2 & -i\hat{\alpha}\hat{\lambda} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -ie^{-i\phi} \end{pmatrix} \\
&= e^{3i\phi}(\lambda - e^{2i\phi}) \begin{pmatrix} 1 & 0 \\ 0 & ie^{i\phi} \end{pmatrix} \begin{pmatrix} i\hat{\alpha}\hat{\lambda} & ie^{-i\phi}(\hat{\beta}^{-1} + \hat{\beta}\hat{\lambda}) \\ \hat{\beta}\hat{\lambda} + \hat{\beta}^{-1}\hat{\lambda}^2 & -e^{-i\phi}\hat{\alpha}\hat{\lambda} \end{pmatrix} \\
&= (e^{3i\phi}\lambda - e^{5i\phi}) \begin{pmatrix} i\hat{\alpha}\hat{\lambda} & ie^{-i\phi}(\hat{\beta}^{-1} + \hat{\beta}\hat{\lambda}) \\ ie^{i\phi}(\hat{\beta}\hat{\lambda} + \hat{\beta}^{-1}\hat{\lambda}^2) & -i\hat{\alpha}\hat{\lambda} \end{pmatrix} \\
&= (e^{3i\phi}\lambda - e^{5i\phi}) \begin{pmatrix} -ie^{2i\phi}\hat{\alpha}\lambda & ie^{-i\phi}\hat{\beta}^{-1} - ie^{i\phi}\hat{\beta}\lambda \\ -ie^{3i\phi}\hat{\beta}\lambda + ie^{5i\phi}\hat{\beta}^{-1}\lambda^2 & ie^{2i\phi}\hat{\alpha}\lambda \end{pmatrix} \\
&= \begin{pmatrix} -ie^{5i\phi}\hat{\alpha}\lambda^2 + ie^{7i\phi}\hat{\alpha}\lambda & ie^{2i\phi}\hat{\beta}^{-1}\lambda - ie^{4i\phi}\hat{\beta}\lambda^2 - ie^{4i\phi}\hat{\beta}^{-1} + ie^{6i\phi}\hat{\beta}\lambda \\ -ie^{6i\phi}\hat{\beta}\lambda^2 + ie^{8i\phi}\hat{\beta}^{-1}\lambda^3 + ie^{8i\phi}\hat{\beta}\lambda - ie^{10i\phi}\hat{\beta}^{-1}\lambda^2 & ie^{5i\phi}\hat{\alpha}\lambda^2 - ie^{7i\phi}\hat{\alpha}\lambda \end{pmatrix} \\
&= \begin{pmatrix} ie^{7i\phi}\hat{\alpha}\lambda - ie^{5i\phi}\hat{\alpha}\lambda^2 & -ie^{4i\phi}\hat{\beta}^{-1} + (ie^{2i\phi}\hat{\beta}^{-1} + ie^{6i\phi}\hat{\beta})\lambda - ie^{4i\phi}\hat{\beta}\lambda^2 \\ ie^{8i\phi}\hat{\beta}\lambda - (ie^{6i\phi}\hat{\beta} + ie^{10i\phi}\hat{\beta}^{-1})\lambda^2 + ie^{8i\phi}\hat{\beta}^{-1}\lambda^3 & -ie^{7i\phi}\hat{\alpha}\lambda + ie^{5i\phi}\hat{\alpha}\lambda^2 \end{pmatrix}.
\end{aligned}$$

The resulting matrix-valued polynomial does not belong to $\mathcal{P}_2$: for example $ie^{8i\phi}\hat{\beta}$ (corresponding to γ of $\zeta_\lambda \in \mathcal{P}_2$) does not belong to $\mathbb{R}^+$ in general.
One possible solution is

$$\zeta_\lambda = -ie^{-3i\phi} \begin{pmatrix} 1 & 0 \\ 0 & ie^{-i\phi} \end{pmatrix} \hat{\zeta}_{\hat{\lambda}} \begin{pmatrix} 1 & 0 \\ 0 & -ie^{i\phi} \end{pmatrix} (\lambda - e^{2i\phi}). \tag{3.2}$$

Direct computation of the right-hand side of 3.2 yields

$$\begin{aligned}
&-ie^{-3i\phi}\begin{pmatrix}1 & 0\\ 0 & ie^{-i\phi}\end{pmatrix}\hat{\zeta}_{\hat{\lambda}}\begin{pmatrix}1 & 0\\ 0 & -ie^{i\phi}\end{pmatrix}(\lambda - e^{2i\phi})\\
&= -ie^{-3i\phi}(\lambda - e^{2i\phi})\begin{pmatrix}1 & 0\\ 0 & ie^{-i\phi}\end{pmatrix}\begin{pmatrix}i\hat{\alpha}\hat{\lambda} & -\hat{\beta}^{-1} - \hat{\beta}\hat{\lambda}\\ \hat{\beta}\hat{\lambda} + \hat{\beta}^{-1}\hat{\lambda}^2 & -i\hat{\alpha}\hat{\lambda}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -ie^{i\phi}\end{pmatrix}\\
&= -ie^{-3i\phi}(\lambda - e^{2i\phi})\begin{pmatrix}1 & 0\\ 0 & ie^{-i\phi}\end{pmatrix}\begin{pmatrix}i\hat{\alpha}\hat{\lambda} & ie^{i\phi}(\hat{\beta}^{-1} + \hat{\beta}\hat{\lambda})\\ \hat{\beta}\hat{\lambda} + \hat{\beta}^{-1}\hat{\lambda}^2 & -e^{i\phi}\hat{\alpha}\hat{\lambda}\end{pmatrix}\\
&= (ie^{-i\phi} - ie^{-3i\phi}\lambda)\begin{pmatrix}i\hat{\alpha}\hat{\lambda} & ie^{i\phi}(\hat{\beta}^{-1} + \hat{\beta}\hat{\lambda})\\ ie^{-i\phi}(\hat{\beta}\hat{\lambda} + \hat{\beta}^{-1}\hat{\lambda}^2) & -i\hat{\alpha}\hat{\lambda}\end{pmatrix}\\
&= (ie^{-i\phi} - ie^{-3i\phi}\lambda)\begin{pmatrix}-ie^{2i\phi}\hat{\alpha}\lambda & ie^{i\phi}\hat{\beta}^{-1} - ie^{3i\phi}\hat{\beta}\lambda\\ -ie^{i\phi}\hat{\beta}\lambda + ie^{3i\phi}\hat{\beta}^{-1}\lambda^2 & ie^{2i\phi}\hat{\alpha}\lambda\end{pmatrix}\\
&= \begin{pmatrix}e^{i\phi}\hat{\alpha}\lambda - e^{-i\phi}\hat{\alpha}\lambda^2 & -\hat{\beta}^{-1} + (e^{-2i\phi}\hat{\beta}^{-1} + e^{2i\phi}\hat{\beta})\lambda - \hat{\beta}\lambda^2\\ \hat{\beta}\lambda - (e^{-2i\phi}\hat{\beta} + e^{2i\phi}\hat{\beta}^{-1})\lambda^2 + \hat{\beta}^{-1}\lambda^3 & -e^{i\phi}\hat{\alpha}\lambda + e^{-i\phi}\hat{\alpha}\lambda^2\end{pmatrix}.
\end{aligned}$$

If we define $\alpha = e^{i\phi}\hat{\alpha} \in \mathbb{C}$, $\beta = e^{-2i\phi}\hat{\beta}^{-1} + e^{2i\phi}\hat{\beta} \in \mathbb{C}$, $\gamma = \hat{\beta} \in \mathbb{R}^+$, then the resulting matrix-valued polynomial belongs to $\mathcal{P}_2$.

Remark: In [2] the proof of theorem 3.2 (iv) transforms the following equations

$$|\lambda_1 + \lambda_1^{-1}| = x^2 + z,\ |\lambda_1 + \lambda_1^{-1}|^2 = 4x^2 + y^2 + z^2,\ (x, y, z) \in \mathbb{R}^3$$

into

$$x^4 - (|\lambda_1 + \lambda_1^{-1}| - 2)x^2 + y^2 = 0 \iff y = \pm x\sqrt{|\lambda_1 + \lambda_1^{-1}| - 2 - x^2}.$$

The correct transformation is

$$\begin{aligned}
&|\lambda_1 + \lambda_1^{-1}|^2 = 4x^2 + y^2 + z^2\\
&= 4x^2 + y^2 + (|\lambda_1 + \lambda_1^{-1}| - x^2)^2\\
&= 4x^2 + y^2 + |\lambda_1 + \lambda_1^{-1}|^2 + x^4 - 2|\lambda_1 + \lambda_1^{-1}|\, x^2\\
&\Leftrightarrow x^4 - (2|\lambda_1 + \lambda_1^{-1}| - 4)x^2 + y^2 = 0\\
&\Leftrightarrow y = \pm x\sqrt{2|\lambda_1 + \lambda_1^{-1}| - 4 - x^2}.
\end{aligned}$$

4 Lattices of Periods

For $a \in \mathcal{M}_2 \setminus \mathcal{M}_2^5$ and initial value $\zeta_\lambda \in I(a)$ we define the set

$$\Gamma_a^\zeta := \{(x,y) \in \mathbb{R}^2 \mid \forall \zeta_\lambda \in I(a) : \phi(x,y)(\zeta_\lambda) = \zeta_\lambda\}.$$

Lemma 4.1. *The set $\Gamma_a^\zeta \subset \mathbb{R}^2$ is an additive, abelian subgroup of $\mathbb{R}^2$. In particular, Γ_a^ζ is a normal subgroup and $\mathbb{R}^2/\Gamma_a^\zeta$ is well-defined.*

Proof.

- Additivity: $\phi(x_1+x_2, y_1+y_2)(\zeta_\lambda) = \phi(x_1,y_1)(\phi(x_2,y_2)(\zeta_\lambda)) = \zeta_\lambda$ for $(x_1,y_1),(x_2,y_2) \in \Gamma_a^\zeta$.
- Neutral element: $\phi(0,0)(\zeta_\lambda) = \zeta_\lambda$.
- Inverse element: $\phi(-x,-y)(\zeta_\lambda) = \phi(-x,-y)(\phi(x,y)(\zeta_\lambda)) = \zeta_\lambda$ for $(x,y) \in \Gamma_a^\zeta$.
- Commutativity: inherited from $\mathbb{R}^2$.

Every subgroup of an abelian group is a normal subgroup, so the second statement follows. □

Observe that the subgroup Γ_a^ζ is independent of the choice of the initial value $\zeta_\lambda \in I(a)$, so we omit the superscript ζ and write Γ_a.

Proof. Let $\zeta, \tilde{\zeta} \in I(a)$. We show $\Gamma_a^\zeta \subset \Gamma_a^{\tilde{\zeta}}$. Consider $(x,y) \in \Gamma_a^\zeta$. By transitivity of the action ϕ there exists $(a,b) \in \mathbb{R}^2$ such that $\phi(a,b)(\tilde{\zeta}) = \zeta$.

$$\begin{aligned}\tilde{\zeta} =& \phi(-a,-b)(\zeta) = \phi(-a,-b)(\phi(x,y)(\zeta)) \\ =& \phi(-a,-b)(\phi(x,y)(\phi(a,b)(\tilde{\zeta}))) = \phi(x,y)(\tilde{\zeta}).\end{aligned}$$

□

We now focus on Γ_a generated by $\mathcal{M}_2^1$.

Definition 4.2. *A subgroup $\Gamma \subset \mathbb{R}^n$ is called **discrete** if there exists an open set $U \subset \mathbb{R}^n$ containing zero such that*

$$U \cap \Gamma = \{0\}.$$

Lemma 4.3. *Let $a \in \mathcal{M}_2^1$. The factor group $\mathbb{R}^2/\Gamma_a$ is compact and the subgroup $\Gamma_a \subset \mathbb{R}^2$ is discrete.*

For proof, see [1].

Remark: Compactness of $\mathbb{R}^2/\Gamma_a$ follows from compactness of $I(a)$.

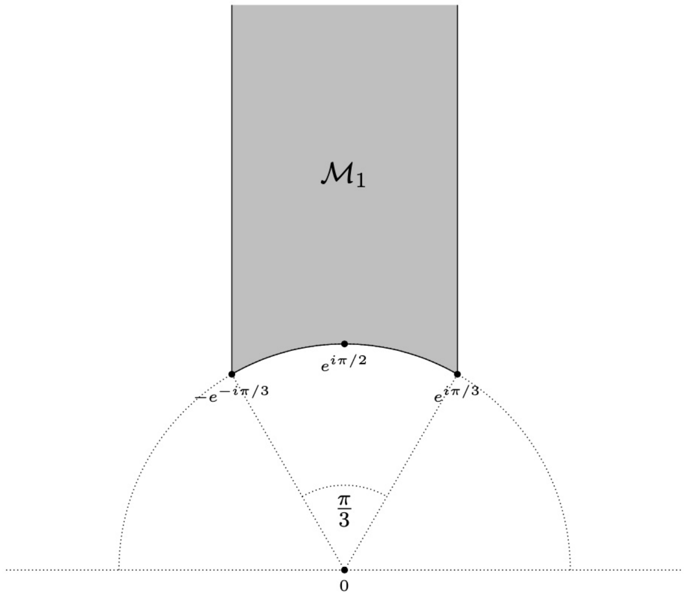

Figure 4.1: Moduli space [3]

The only discrete subgroups of $\mathbb{R}^2$ are the following ones:

1. $\Gamma_a = \{0\}$. In this case $\mathbb{R}^2/\Gamma_a \cong \mathbb{R}^2$, which contradicts the compactness condition.
2. $\Gamma_a = \omega\mathbb{Z}$. In this case $\mathbb{R}^2/\Gamma_a \cong \mathbb{R} \times \mathbb{S}^1$, which again contradicts the compactness condition. For example, consider the map

$$\mathbb{R} \times \mathbb{S}^1 \to \mathbb{C}/\omega\mathbb{Z},\ (s, e^{2\pi it}) \mapsto (t + is)\omega.$$

3. $\Gamma_a = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, where ω_1, ω_2 are linearly independent vectors in $\mathbb{R}^2$. In this case $\mathbb{R}^2/\Gamma_a \cong \mathbb{S}^1 \times \mathbb{S}^1$, which is a compact torus. For example, consider the map

$$\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{C}/\Gamma_a,\ (e^{2\pi it}, e^{2\pi is}) \mapsto t\omega_1 + s\omega_2.$$

Hence, Γ_a is a lattice generated by $\omega_1, \omega_2 \in \mathbb{C}$. The choice of these generators is not unique.

Definition 4.4. *We call two lattices **isomorphic** if they originate from one another through a rotation-dilation.*

Theorem 4.5. *Each lattice Γ is isomorphic to $\Gamma_\tau := \mathbb{Z} + \mathbb{Z}\tau$ with a unique τ in*

$$\mathcal{M}_1 := \left\{\tau \in \mathbb{C} \mid \Im(\tau) > 0, |\Re(\tau)| \leq \frac{1}{2}, |\tau| \geq 1\right\} / \sim,$$

*where the equivalence relation identifies $\tau \in \partial\mathcal{M}_1$ with $-\overline{\tau}$. The set $\mathcal{M}_1$ is called the **moduli space**.*

Proof can be found in [3].

As a result, we can identify each $a \in \mathcal{M}_2^1$ with an element from $\mathcal{M}_1$, i.e. the following map is well-defined:

$$g : \mathcal{M}_2^1 \to \mathcal{M}_1, \ a \mapsto \tau_a.$$

We want to prove that g has a unique surjective continuous extension to $\mathcal{M}_2^2 \cup \mathcal{M}_2^3$.

The rest of this section is a brief description of the strategy behind the proof of the foregoing statement. For more details, see [2].

For $a \in \mathcal{M}_2^1$ we introduce the monodromy M_ω for $\omega \in \Gamma_a$.

Definition 4.6. *Let $\zeta_\lambda : \mathbb{R}^2 \to \mathcal{P}_2$ be a Polynomial Killing field with initial potential $\zeta_0 \in \mathcal{P}_2$. The fundamental solution F of the following system of ODEs is called **frame**:*

$$\frac{\partial F}{\partial x} = FU(\zeta_\lambda), \ \frac{\partial F}{\partial y} = FV(\zeta_\lambda), \ F(0,0) = \mathbb{1}.$$

*The **monodromy** M_ω is defined as the value of F at ω:*

$$F(\omega) = M_\omega, \ \omega \in \Gamma_a.$$

Lemma 4.7. *The monodromies M_ω and the initial value of a Polynomial Killing Field $\zeta_\lambda^0 = \zeta_\lambda(0,0)$ commute pairwise.*

For proof, see [1].

Lemma 4.8. *The monodromies satisfy* $\det(M_\omega) = 1$.

For proof, see [1].

We look for eigenvalues $\nu, -\nu$ of ζ_λ (recall the fact that $tr(\zeta_\lambda) = 0$). We can put ν in concrete terms:

$$0 = \det(\zeta_\lambda - \nu\mathbb{1}) = \nu^2 + \lambda a(\lambda).$$

The smooth Riemann surface

$$\Sigma^* = \{(\lambda, \nu) \in \mathbb{C}^* \times \mathbb{C} \mid \nu^2 + \lambda a(\lambda) = 0\}$$

parametrizes the one-dimensional eigenspaces of ζ_0.

Consequence: M_ω maps the eigenspaces of ζ_λ^0 into themselves, so M_ω acts on the one-dimensional eigenspaces of ζ_λ^0 as the multiplication with a function $\mu_\omega : \Sigma^* \to \mathbb{C} \setminus \{0\}$. The reality condition $\overline{\lambda}^4 a(\overline{\lambda}^{-1}) = \overline{a(\lambda)}$ induces the second of the following involutions:

$$\sigma : (\lambda, \nu) \mapsto (\lambda, -\nu), \quad \rho : (\lambda, \nu) \mapsto (\overline{\lambda}^{-1}, -\overline{\lambda}^{-3}\overline{\nu}).$$

These involutions act on μ_ω as

$$\sigma^* \mu_\omega = \mu_\omega^{-1}, \quad \rho^* \mu_\omega = \overline{\mu}_\omega^{-1}. \tag{4.1}$$

We can define

$$d\ln\mu_\omega = \frac{b_\omega(\lambda)}{2\nu} d\ln\lambda,\ \ b_\omega \in \mathbb{C}^3[\lambda].$$

Due to Theorem 3.2, the groups Γ_a are no lattices for $a \in \mathcal{M}_2^2 \cup \mathcal{M}_2^3 \cup \mathcal{M}_2^4 \cup \mathcal{M}_2^5$. For $a \in \mathcal{M}_2^2 \cup \mathcal{M}_2^3$ we impose in addition to condition 4.1

$$\mu_\omega = f_\omega(\lambda) + g_\omega(\lambda)\nu \ \ \text{with holomorphic } f_\omega, g_\omega : \mathbb{C} \setminus \{0\} \to \mathbb{C}. \tag{4.2}$$

This allows us to prove the following result:

Theorem 4.9. *For $a \in \mathcal{M}_2^1 \cup \mathcal{M}_2^2 \cup \mathcal{M}_2^3$ the values $b_\omega(0)$ build a lattice $\tilde{\Gamma}_a$ in $\mathbb{C}$: they define $d\ln\mu_\omega$ of a function μ_ω on Σ^* which obeys 4.1 and 4.2. The map $g : \mathcal{M}_2^1 \cup \mathcal{M}_2^2 \cup \mathcal{M}_2^3 \to \mathcal{M}_1$ to the corresponding isomorphism class is continuous. For all compact subsets $\mathcal{K} \subset \mathcal{M}_1$ each $a \in \mathcal{M}_2^4 \cup \mathcal{M}_2^5$ has a neighborhood O in $\mathcal{M}_2$ such that $g^{-1}[\mathcal{K}] \cap O = \emptyset$.*

Remark: The last statement ensures that the map g extends to a continuous map from $\mathcal{M}_2$ to $\mathcal{M}_1 \cup \{\infty\}$, such that g takes the value ∞ on $\mathcal{M}_2^4 \cup \mathcal{M}_2^5$.

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