Stochastic Calculus
Tutorial 5

Itô processes

Fix $T \in (0,\infty)$. Let $B = (B_t)_{t\in[0,T]}$ be a Brownian motion on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_t)_{t\in[0,T]}$ the corresponding Brownian standard filtration (satifying the usual conditions).

Definition. A stochastic process is called *Itô process* if there is a ($\mathbb{P}$-a.s) representation

$$X_t = X_0 + \int_0^t a(\cdot,s)\mathrm{d}s + \int_0^t b(\cdot,s)\mathrm{d}B_s \quad t \in [0,T],$$

where $X_0 \in \mathbb{R}$ and $a, b : \Omega \times [0,T] \to \mathbb{R}$ are adapted, measurable processes satisfying the integrability conditions

$$\mathbb{P}\left(\int_0^T |a(\omega,s)|\mathrm{d}s < \infty\right) = 1 \quad \text{and} \quad \mathbb{P}\left(\int_0^T |b(\omega,s)|^2\mathrm{d}s < \infty\right) = 1.$$

Remark. (i) Note that $b \in H^2_{loc}$, thus the stochastic integral is well-defined.

(ii) If we decompose $X = M + A$ with $M_t := \int_0^t b(\cdot,s)\mathrm{d}B_s$, $A_t := X_0 + \int_0^t a(\cdot,s)\mathrm{d}s$, $t \in [0,T]$, we have that M is a local martingale and A has finite variation.

Lemma. *Let $X = (X_t)_{t\in[0,T]}$ be an Itô process with representations*

$$X_t = X_0 + \int_0^t a(\cdot,s)\mathrm{d}s + \int_0^t b(\cdot,s)\mathrm{d}B_s = \tilde{X}_0 + \int_0^t \tilde{a}(\cdot,s)\mathrm{d}s + \int_0^t \tilde{b}(\cdot,s)\mathrm{d}B_s$$

for $t \in [0,T]$ and $X_0 = \tilde{X}_0$. Then, we have $a = \tilde{a}$ and $b = \tilde{b}$, $\mathbb{P} \otimes \lambda$-a.s.

Itô's integration for Itô processes

Proposition. *For any Itô process $X = (X_t)_{t\in[0,T]}$ with representation*

$$X_t = X_0 + \int_0^t a(\cdot,s)\mathrm{d}s + \int_0^t b(\cdot,s)\mathrm{d}B_s, \quad t \in [0,T],$$

the quadratic variation of X is given by

$$\langle X\rangle_t := \lim_{n\to\infty} \sum_{J\in\Pi_n} \left(\Delta_{J\cap[0,t]}X\right)^2 = \int_0^t b^2(\cdot,s)\mathrm{d}s, \quad t \in [0,T],$$

for any zero-sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$.

Definition. Let $X = (X_t)_{t\in[0,T]}$ be an Itô process with representation $X_t = X_0 + \int_0^t a(\cdot,s)\mathrm{d}s + \int_0^t b(\cdot,s)\mathrm{d}B_s$, $t \in [0,T]$. We introduce

$$\mathcal{L}(X) := \left\{ f : \Omega \times [0,T] \to \mathbb{R} : \substack{f \text{ is measurable, adapted and} \\ \int_0^T |f(\cdot,s)a(\cdot,s)|\mathrm{d}s<\infty \text{ and } \int_0^T |f(\cdot,s)b(\cdot,s)|^2\mathrm{d}s<\infty\, \mathbb{P}\text{-a.s.}} \right\}.$$

For $f \in \mathcal{L}(X)$ we define the *stochastic Itô integral* by

$$\int_0^t f(\cdot,s)\mathrm{d}X_s := \int_0^t f(\cdot,s)a(\cdot,s)\mathrm{d}s + \int_0^t f(\cdot,s)b(\cdot,s)\mathrm{d}B_s, \quad t \in [0,T].$$

Remark. Note that $fb \in H^2_{loc}$, thus the stochastic integral is well-defined .

Lemma. *Let $X = (X_t)_{t\in[0,T]}$ be an Itô process with representation $X_t = X_0 + \int_0^t b(\cdot,s)\mathrm{d}B_s$, $t \in [0,T]$, i.e. $a = 0$, and let $f \in \mathcal{L}(X)$.*

(i) The integral process $(\int_0^t f(\cdot,s)\mathrm{d}X_s)_{t\in[0,T]}$ is a continuous local martingale.

(ii) If $\mathbb{E}[\int_0^T f^2(\cdot,s)b^2(\cdot,s)\mathrm{d}s] < \infty$, then Itô's isometry holds true:

$$\mathbb{E}\left[\left(\int_0^t f(\cdot,s)\mathrm{d}X_s\right)^2\right] = \mathbb{E}\left[\int_0^t f^2(\cdot,s)\mathrm{d}\langle X\rangle_s\right], \quad t \in [0,T].$$

Remark. Since $\langle X\rangle_s$ is a non-decreasing process, $\int_0^t f^2(\cdot,s)\mathrm{d}\langle X\rangle_s$ is just a Lebesgue-Stieltjes integral and by the associativity of the Riemann integral we have

$$\int_0^t f^2(\cdot,s)\mathrm{d}\langle X\rangle_s = \int_0^t f^2(\cdot,s)b^2(\cdot,s)\mathrm{d}s, \quad t \in [0,T].$$

Itô's formula for Itô processes

Lemma. *Let $g \in C(\mathbb{R}^2)$, $(Z_t)_{t\in[0,T]}$ be a continuous and adapted stochastic process and $(X_t)_{t\in[0,T]}$ be an Itô process. For any $t \in [0,T]$ and $t_i := \frac{i}{n}t$, $i = 0,\dots,n$, we have*

$$\sum_{i=1}^n g(Z_{t_{i-1}})(X_{t_i} - X_{t_{i-1}})^2 \overset{\mathbb{P}}{\to} \int_0^t g(Z_s)\mathrm{d}\langle X\rangle_s \quad \text{for } n \to \infty.$$

Definition. Let X, Y be two Itô processes. Then $X+Y$ and $X-Y$ are also Itô processes and we define the *co-variation process* of X and Y by

$$\langle X,Y\rangle_t := \frac{1}{4}\left(\langle X+Y\rangle_t - \langle X-Y\rangle_t\right), \quad t \in [0,T].$$

Theorem (Itô's formula for Itô processes)**.** *Let $f \in C^2(\mathbb{R}^2)$ and X, Y be two Itô processes with representations*

$$X_t = X_0 + \int_0^t a(\cdot,s)\mathrm{d}s + \int_0^t b(\cdot,s)\mathrm{d}B_s \quad \text{and} \quad Y_t = Y_0 + \int_0^t \alpha(\cdot,s)\mathrm{d}s + \int_0^t \beta(\cdot,s)\mathrm{d}B_s,$$

for $t \in [0,T]$, $\mathbb{P}$-a.s. Then, we have that

$$\begin{aligned} f(t,X_t) = f(X_0,Y_0) &+ \int_0^t f_x(X_s,Y_s)\mathrm{d}X_s + \int_0^t f_y(X_s,Y_s)\mathrm{d}Y_s \\ &+ \frac{1}{2}\int_0^t f_{xx}(X_s,Y_s)\mathrm{d}\langle X\rangle_s + \frac{1}{2}\int_0^t f_{yy}(X_s,Y_s)\mathrm{d}\langle Y\rangle_s \\ &+ \int_0^t f_{xy}(X_s,Y_s)\mathrm{d}\langle X,Y\rangle_s, \end{aligned}$$

for $t \in [0,T]$, $\mathbb{P}$-a.s.

Hints for Exercise Sheet 6

Exercise 6.1
Apply the product rule.
Exercise 6.2
Existence: Use Itô's formula.
Uniqueness: For two solutions Z, Z' show that $\frac{Z}{Z'} = 1$ by, first, using Itô's formula for $g : (x,y) \mapsto g(x,y) = \exp(-(x - \frac{1}{2}y))$ and then, applying the product rule.
Exercise 6.3
Further, use that if M is a martingale, it holds $\mathbb{E}^{x_0}[M_{t_0}] = \mathbb{E}^{x_0}[M_0]$, $t_0 \in (0,T]$.