Stochastic Calculus
Tutorial 4

Stochastic Itô integration - extension of Itô integration via localization

Fix $T \in (0,\infty)$. Let $B = (B_t)_{t\in[0,T]}$ be a Brownian motion on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_t)_{t\in[0,T]}$ the corresponding Brownian standard filtration (satifying the usual conditions).

Definition. We introduce

$$\mathscr{H}^2_{loc} := \left\{ f : \Omega \times [0,T] \to \mathbb{R} : f \text{ is measurable, adapted and } \int_0^T f^2(\cdot, s)\mathrm{d}s < \infty \right\}.$$

Definition. An increasing sequence $(\nu_n)_{n\in\mathbb{N}}$ of $[0,T]$-valued stopping times is called *localizing sequence* for $f \in \mathscr{H}^2_{loc}$ if $f\mathbb{1}_{[0,\nu_n]} \in \mathscr{H}^2$ for all $n \in \mathbb{N}$ and $\mathbb{P}\left(\bigcup_{n\in\mathbb{N}} \{\nu_n = T\}\right) = 1$.

Remark. (i) $\mathscr{H}^2 \subseteq \mathscr{H}^2_{loc}$.

(ii) For any continuous $g : \mathbb{R} \to \mathbb{R}$ it holds that $f(\omega, t) = g(B_t(\omega)) \in \mathscr{H}^2_{loc}$ since B is a.s. pathwise continuous.

Proposition. *For every $f \in \mathscr{H}^2_{loc}$ there exists a localizing sequence $(\nu_n)_{n\in\mathbb{N}}$.*

Definition. Let $f \in \mathscr{H}^2_{loc}$ and $(\nu_n)_{n\in\mathbb{N}}$ be a localizing sequence for f.
The *Itô integral process* $(\int_0^t f(\cdot,s)\mathrm{d}B_s)_{t\in[0,T]}$ is defined as the continuous process $X = (X_t)_{t\in[0,T]}$ such that

$$\int_0^t f(\cdot, s)dB_s := X_t = \lim_{n\to\infty} \int_0^t f\mathbb{1}_{[0,\nu_n]}dB_s \quad \mathbb{P}\text{-a.s.}$$

for all $t \in [0,T]$.

Remark. Recall that $f\mathbb{1}_{[0,\nu_n]} \in \mathscr{H}^2$, $n \in \mathbb{N}$.

Theorem. *For $f \in \mathscr{H}^2_{loc}$ there exists a continuous local martingale $(X_t)_{t\in[0,T]}$ such that for any localizing sequence $(\nu_n)_{n\in\mathbb{N}}$ of f it holds that*

$$\int_0^t f(\cdot, s)\mathbb{1}_{[0,\nu_n]}\mathrm{d}B_s \to X_t \quad \text{as } n \to \infty \quad \mathbb{P}\text{-a.s.},$$

for all $t \in [0,T]$. In particular, $(X_t)_{t\in[0,T]}$ does not depend on the choice of the localizing sequence $(\nu_n)_{n\in\mathbb{N}}$, and the Itô integral process $(\int_0^t f(\cdot,s)\mathrm{d}B_s)_{t\in[0,T]}$ is well-defined.

Theorem (Persistence of identity)**.** *Let $f, g \in \mathscr{H}^2_{loc}$ and ν be a stopping time such that $f\mathbb{1}_{[0,\nu]} = g\mathbb{1}_{[0,\nu]}$. Then it holds*

$$\int_0^t f(\cdot, s)\mathrm{d}B_s\mathbb{1}_{[0,\nu]} = \int_0^t g(\cdot, s)\mathrm{d}B_s\mathbb{1}_{[0,\nu]} \quad \mathbb{P}\text{-a.s.}$$

for all $t \in [0,T]$.

Theorem (Riemann sum approximation)**.** *If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $t_i = \frac{i}{n}T$, $i = 0, \dots, n$, then, for $n \to \infty$, we have*

$$\sum_{i=1}^n f(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) \to \int_0^T f(B_s)\mathrm{d}B_s \quad \text{in probability.}$$

Stochastic Itô integration - Itô formula for Brownian motion

Theorem (Itô formula). *For any twice continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ we have*

$$f(B_t) = f(0) + \int_0^t f'(B_s)\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(B_s)\mathrm{d}B_s,$$

for $t \in [0,T]$, $\mathbb{P}$-a.s.

Remark. We denote by $C^{1,2}([0,T]\times\mathbb{R})$ the space of continuous functions $f : [0,T]\times\mathbb{R} \to \mathbb{R}, (t,x) \mapsto f(t,x)$ such that $f(t,x)$ is continuously differentiable in $t \in (0,T)$ and twice continuously differentiable in $x \in \mathbb{R}$.

Theorem (Itô formula, space-time version). *For any $f \in C^{1,2}([0,T]\times\mathbb{R})$ we have*

$$f(t,B_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial t}(s,B_s)\mathrm{d}B_s + \int_0^t \frac{\partial f}{\partial x}(s,B_s)\mathrm{d}B_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,B_s)\mathrm{d}s,$$

for $t \in [0,T]$, $\mathbb{P}$-a.s.

Hints for Exercise Sheet 5

Exercise 5.1

(i) We want to apply Lemma 3.19.

(ii) Use Itô's formula.

Exercise 5.2

(i) Apply the fundamental theorem of calculus.

(iii) Use Itô's formula.

Exercise 5.3

(i) Use the definition of $\langle X, Y\rangle_t$.

(i) Use Itô's formula for $f(X_t)$ and $g(Y_t)$ and work again with the definition of $\langle f(X), g(Y)\rangle_t$