Stochastic Calculus
Tutorial 3

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Stochastic Itô integration - construction of the Itô integral

Fix $T \in (0, \infty)$. Let $B = (B_t)_{t\in[0,T]}$ be a Brownian motion on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_t)_{t\in[0,T]}$ the corresponding Brownian standard filtration (satifying the usual conditions).

Definition. A function $f : \Omega \times [0,T] \to \mathbb{R}$ is called *simple function* if there exist real numbers $0 = t_0 < t_1 < \dots t_n = T$ and $\mathcal{F}_{t_i}$-measurable random variables a_i with $\mathbb{E}[a_i^2] < \infty$, $i = 0, \dots, n-1$, $n \in \mathbb{N}$ such that

$$f(\omega, s) = \sum_{i=0}^{n-1} a_i(\omega) \mathbb{1}_{(t_i,t_{i+1}](s)}$$

for $\omega \in \Omega$, $s \in [0,T]$.
We write $\mathscr{H}_0^2$ for the set of simple functions.

Definition (Stochastic integral for simple functions)**.** For $f \in \mathscr{H}_0^2$ we define

$$I(f) := \sum_{i=0}^{n-1} a_i (B_{t_{i+1}} - B_{t_i}) := \int_0^T f(\cdot, s) \mathrm{d}B_s.$$

We consider the space $\mathscr{H}_0^2$ as a subspace of $L^2(\Omega \times [0,T], \mathbb{P} \otimes \lambda)$ with the norm

$$\|f\|_{\mathscr{H}^2} := \|f\|_{L^2(\mathbb{P}\otimes\lambda)} := \mathbb{E}\left[\int_0^T f^2(\cdot, s)\mathrm{d}s\right]^{\frac{1}{2}} \quad \text{for } f \in L^2(\Omega \times [0,T], \mathbb{P} \otimes \lambda).$$

Lemma (Itô's isometry for simple functions)**.** *For $f \in \mathscr{H}_0^2$ it holds $\|f\|_{\mathscr{H}^2} = \|I(f)\|_{L^2}$.*

Remark. For $f \in \mathscr{H}_0^2$ it holds $\mathbb{E}\left[\int_0^T f^2(\cdot, s)\mathrm{d}\langle B\rangle_s\right] = \mathbb{E}[(\int_0^T f(\cdot, s)\mathrm{d}B_s)^2]$.

Definition. (i) $f : \Omega \times [0,T] \to \mathbb{R}$ is called *measurable* if f is $(\mathcal{F} \otimes \mathcal{B}([0,T]), \mathcal{B}(\mathbb{R}))$-measurable.

(ii) $f : \Omega \times [0,T] \to \mathbb{R}$ is called *adapted* if $f(\cdot, t)$ is $\mathcal{F}_t$-measurable for all $t \in [0,T]$.

(iii) The set of all adapted functions in $L^2(\Omega \times [0,T], \mathbb{P} \otimes \lambda)$ is given by

$$\mathscr{H}^2 := \left\{ f : \Omega \times [0,T] \to \mathbb{R} : f \text{ is measurable, adapted and } \mathbb{E}\left[\int_0^T f^2(\cdot, s)\mathrm{d}s\right] < \infty \right\}.$$

Proposition. *For every $f \in \mathscr{H}^2$ there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathscr{H}_0^2$ such that*

$$\|f_n - f\|_{\mathscr{H}^2} \to 0 \quad \text{as } n \to \infty.$$

Definition. We define $I : \mathscr{H}^2 \to L^2$ via

$$I(f) := \lim_{n\to\infty} I(f_n) \quad \text{for } f \in \mathscr{H}^2$$

where we take $\|I(f_n) - I(f)\|_{L^2} \to 0$ for any sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathscr{H}_0^2$ such that $\|f_n - f\|_{\mathscr{H}^2} \to 0$ as $n \to \infty$.

Theorem (Itô's isometry)**.** *For $f \in \mathscr{H}^2$ it holds $\|f\|_{\mathscr{H}^2} = \|I(f)\|_{L^2}$.*

Remark. Note that for $f \in \mathscr{H}^2$ we also have that $f\mathbb{1}_{[0,t]} \in \mathscr{H}^2$.

Theorem. *For $f \in \mathscr{H}^2$ there is a continuous martingale $X = (X_t)_{t\in[0,T]}$ w.r.t. $(\mathcal{F}_t)_{t\in[0,T]}$ such that for all $t \in [0,T]$:*

$$X_t = I(f\mathbb{1}_{[0,t]}) \quad \mathbb{P}\text{-a.s.}$$

Note that X is adapted and $(\mathcal{F} \otimes \mathcal{B}([0,T]))$-measurable.

Definition. For $f \in \mathscr{H}^2$ the *Itô integral* (or the *integral process*) is defined by

$$\int_0^t f(\cdot, s) \mathrm{d}B_s := X_t := I(f \mathbb{1}_{[0,t]}) \quad \mathbb{P}\text{-a.s.}, \quad t \in [0,T],$$

where $X = (X_t)_{[0,T]}$ is defined as in the theorem above.

Remark. The stochastic integral $\int_0^t f(\cdot, s) \mathrm{d}B_s$ is a linear operator on $\mathscr{H}^2$ since it is a linear operator on $\mathscr{H}_0^2$.

Proposition. *Let $f \in \mathscr{H}^2$ and ν be a stopping time satisfying $f \mathbb{1}_{[0,\nu]} = 0$. The integral proces $X = (X_t)_{t \in [0,T]}$, with $X_t = \int_0^t f(\cdot, s) \mathrm{d}B_s$, then fulfills $X \mathbb{1}_{[0,\nu]} = 0$. In particular, for two functions $f, g \in \mathscr{H}^2$ with $f \mathbb{1}_{[0,\nu]} = g \mathbb{1}_{[0,\nu]}$ the integral process coincide on $[0, \nu]$.*