HWS 2021
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Stochastic Calculus
Tutorial 2

Martingales

Lemma 1. *Consider a real-valued stochastic process $(X_t)_{t\in[0,T]}$ adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ with the property that $\mathbb{E}[|X_t|] < \infty$ for all $t \in [0,T]$.*

(i) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function. If $(X_t)_{t\in[0,T]}$ is a martingale with $\mathbb{E}[|\varphi(X_t)|] < \infty$ for $t \in [0,T]$, then $(\varphi(X_t))_{t\in[0,T]}$ is a sub-martingale.

(ii) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a non-decreasing convex function. If $(X_t)_{t\in[0,T]}$ is a sub-martingale with $\mathbb{E}[|\varphi(X_t)|] < \infty$ for $t \in [0,T]$, then $(\varphi(X_t))_{t\in[0,T]}$ is a sub-martingale.

Recall *Jensen's inequality for conditional expectations*: Let $I \subseteq \mathbb{R}$ be an interval, let $\varphi : I \to \mathbb{R}$ be a convex function and let X be an I-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. If $\mathbb{E}[|X|] < \infty$ and $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra, then

$$\varphi(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[\varphi(X)|\mathcal{G}] \leq \infty.$$

Proof. (i) Since $(X_t)_{t\in[0,T]}$ is $(\mathcal{F}_t)$-adapted and φ is measurable as a convex function, we first note that $(\varphi(X_t))_{t\in[0,T]}$ is $(\mathcal{F}_t)$-adapted. By assumption, it holds $\mathbb{E}[|\varphi(X_t)|] < \infty$ for $t \in [0,T]$. For the martingale property, let $s,t \in [0,T]$, $s < t$. Then

$$\mathbb{E}[\varphi(X_t)|\mathcal{F}_s] \overset{\text{Jensen's ineq.}}{\geq} \varphi(\underbrace{\mathbb{E}[X_t|\mathcal{F}_s]}_{\substack{=X_s \text{ a.s.,} \\ (X_t)_t \text{ mart.}}}) = \varphi(X_s) \quad \text{a.s.}$$

(ii) Since $(X_t)_{t\in[0,T]}$ is $(\mathcal{F}_t)$-adapted and φ is measurable as a convex function, we first note that $(\varphi(X_t))_{t\in[0,T]}$ is $(\mathcal{F}_t)$-adapted. By assumption, it holds $\mathbb{E}[|\varphi(X_t)|] < \infty$ for $t \in [0,T]$. For the martingale property, let $s,t \in [0,T]$, $s < t$. Then

$$\mathbb{E}[\varphi(X_t)|\mathcal{F}_s] \overset{\text{Jensen's ineq.}}{\geq} \varphi(\underbrace{\mathbb{E}[X_t|\mathcal{F}_s]}_{\substack{\geq X_s \text{ a.s.,} \\ (X_t)_t \text{ sub-m.}}}) \overset{\varphi \text{ non-decr.}}{\geq} \varphi(X_s) \quad \text{a.s.}$$

□

Lemma 2. *Consider a real-valued integrable random variable X on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. Let $(X_t)_{t\in[0,T]} = (\mathbb{E}[X|\mathcal{F}_t])_{t\in[0,T]}$. Then $(X_t)_{t\in[0,T]}$ is a martingale.*

Proof. By definition of the conditional expectation, $\mathbb{E}[X|\mathcal{F}_t]$ is $\mathcal{F}_t$-measurable for all $t \in [0,T]$, i.e. $(X_t)_{t\in[0,T]}$ is $(\mathcal{F}_t)$-adapted. For $t \in [0,T]$ it holds

$$\mathbb{E}[|X_t|] = \mathbb{E}\big[|\mathbb{E}[X|\mathcal{F}_t]|\big] \underset{\varphi(u)=|u|}{\overset{\text{Jensen's ineq.}}{\leq}} \mathbb{E}\big[\mathbb{E}[|X|\,|\mathcal{F}_t]\big] = \mathbb{E}[|X|] < \infty$$

by assumption. For the martingale property, let $s,t \in [0,T]$, $s < t$. Then

$$\mathbb{E}[X_t|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_t]|\mathcal{F}_s] \underset{\mathcal{F}_s \subset \mathcal{F}_t}{\overset{\text{tower prop.}}{=}} \mathbb{E}[X|\mathcal{F}_s] = X_s \quad \text{a.s.}$$

□

Lemma 3. *Consider two real-valued stochastic processes $(X_t)_{t\in[0,T]}$, $(Y_t)_{t\in[0,T]}$ adapted to the same filtration $(\mathcal{F}_t)_{t\in[0,T]}$ with $\mathbb{E}[|X_t|] < \infty$ and $\mathbb{E}[|Y_t|] < \infty$ for $t \in [0,T]$. Then, if $(X_t)_{t\in[0,T]}$ and $(Y_t)_{t\in[0,T]}$ are both super-martingales, then $(Z_t)_{t\in[0,T]}$ with $Z_t := \min(X_t, Y_t)$, $t \in [0,T]$, is a super-martingale with respect to $(\mathcal{F}_t)_{t\in[0,T]}$.*

Proof. Since $(X_t)_{t\in[0,T]}$ and $(Y_t)_{t\in[0,T]}$ are $(\mathcal{F}_t)$-adapted and since $\min : \mathbb{R}^2 \to \mathbb{R}$ is measurable, $(Z_t)_{t\in[0,T]}$ is $(\mathcal{F}_t)$-adapted. And for $t \in [0,T]$ it holds

$$\mathbb{E}[|Z_t|] \leq \mathbb{E}[|X_t|] + \mathbb{E}[|Y_t|] < \infty$$

by assumption. For the martingale property, let $s,t \in [0,T]$, $s < t$. Then

$$\mathbb{E}[Z_t|\mathcal{F}_s] = \mathbb{E}[\min(X_t,Y_t)|\mathcal{F}_s] \leq \mathbb{E}[X_t|\mathcal{F}_s] \overset{(X_t)_t \text{ super-m.}}{\leq} X_s \quad \text{a.s.}$$

and, analogously,

$$\mathbb{E}[Z_t|\mathcal{F}_s] = \mathbb{E}[\min(X_t,Y_t)|\mathcal{F}_s] \leq \mathbb{E}[Y_t|\mathcal{F}_s] \overset{(Y_t)_t \text{ super-m.}}{\leq} Y_s \quad \text{a.s.,}$$

hence, $\mathbb{E}[Z_t|\mathcal{F}_s] \leq \min(X_s,Y_s) = Z_s$ a.s. □

Hints for Exercise Sheet 3

Exercise 3.1
First, download the latest version of the Exercise Sheet, unfortunately, there was a small mistake in the definition of uniform integrability.

(i) Assume that the random variables $|X_n|$ are uniformly bounded by a finite constant c and show the claim.
Then consider the function $\phi_c(x) = (x \wedge c) \vee (-c)$ for some c. Use Fatou's lemma to show that X is integrable. Apply the triangular inequality twice to show $\mathbb{E}[|X_n - X|] \to 0$ as $n \to \infty$.

Exercise 3.2

(i) Use Exercise 2.3.(ii)

(ii) (a) Use Theorem 2.20.
(b) Apply Doob's L^p-inequality.

Exercise 3.3
Find an approximating sequence of simple functions f_n such that $\|f - f_n\|_2^2 \to 0$ as $n \to \infty$ for $f = B$ and calculate $I(f\mathbb{1}_{[0,t]}) = \lim_{n\to\infty} I(f_n\mathbb{1}_{[0,t]})$.