Stochastic Calculus
Solution Sheet 5

Exercise 5.1

(i) Let $(X_t)_{t\in[0,T]}$ be an Itô process with representations

$$X_t = X_0 + \int_0^t a(\cdot,s)\,\mathrm{d}s + \int_0^t b(\cdot,s)\,\mathrm{d}B_s = \tilde{X}_0 + \int_0^t \tilde{a}(\cdot,s)\,\mathrm{d}s + \int_0^t \tilde{b}(\cdot,s)\,\mathrm{d}B_s$$

for $t\in[0,T]$ and $X_0=\tilde{X}_0$ $\mathbb{P}$-a.s. Hence,

$$Y = \int_0^\cdot \big(b(\cdot,s)-\tilde{b}(\cdot,s)\big)\,\mathrm{d}B_s = \int_0^\cdot \big(\tilde{a}(\cdot,s)-a(\cdot,s)\big)\,\mathrm{d}s$$

is a continuous local martingale of finite variation, as the left-hand side is a local martingale and the right-hand side is of finite variation. Consequently, we know by Lemma 2.19. that

$$Y_t = 0, \quad t\in[0,T], \quad \mathbb{P}\text{-a.s.}$$

Particularly,

$$\int_0^t \big(b(\cdot,s)-\tilde{b}(\cdot,s)\big)\,\mathrm{d}B_s = 0, \quad t\in[0,T], \quad \mathbb{P}\text{-a.s.}$$

By Itô isometry we obtain

$$0 = \mathbb{E}\Bigg[\bigg(\int_0^t \big(b(\cdot,s)-\tilde{b}(\cdot,s)\big)\,\mathrm{d}B_s\bigg)^2\Bigg] = \mathbb{E}\bigg[\int_0^t \big(b(\cdot,s)-\tilde{b}(\cdot,s)\big)^2\,\mathrm{d}s\bigg] = \int_0^t \mathbb{E}\big[\underbrace{\big(b(\cdot,s)-\tilde{b}(\cdot,s)\big)^2}_{\geq 0}\big]\,\mathrm{d}s, \quad t\in[0,T]$$

which implies that

$$b-\tilde{b} = 0 \quad \mathbb{P}\otimes\lambda\text{-a.s.}$$

Furthermore, we know

$$\int_0^t \big(\tilde{a}(\cdot,s)-a(\cdot,s)\big)\,\mathrm{d}s = 0, \quad t\in[0,T], \quad \mathbb{P}\text{-a.s.}$$

which reveals

$$a-\tilde{a} = 0 \quad \mathbb{P}\otimes\lambda\text{-a.s.}$$

(ii) Let X,Y be two Itô processes and $f(x,y):=xy$, $f\in C^2(\mathbb{R}^2)$. Then, $X_tY_t = f(X_t,Y_t)$. We compute

$$\frac{\partial f}{\partial x}(x,y) = y, \quad \frac{\partial f}{\partial y}(x,y) = x, \quad \frac{\partial^2 f}{\partial y\partial x}(x,y) = \frac{\partial^2 f}{\partial x\partial y}(x,y) = 1$$

and

$$\frac{\partial^2 f}{\partial y^2}(x,y) = 0, \quad \frac{\partial^2 f}{\partial x^2}(x,y) = 0.$$

By Itô's formula for Itô processes we get

$$X_tY_t = X_0Y_0 + \int_0^t Y_s\,\mathrm{d}X_s + \int_0^t X_s\,\mathrm{d}Y_s + \langle X,Y\rangle_t, \quad t\in[0,T],$$

noting that $\int_0^t 1\,\mathrm{d}\langle X,Y\rangle_s = \langle X,Y\rangle_t$.

Exercise 5.2

(i) For $f \in C^1(\mathbb{R})$ and $t \mapsto X_t^n \in C^1$, we have $t \mapsto F(X_t^n) \in C^1$, where $F(x) := \int_0^x f(y)\,\mathrm{d}y$. By the fundamental theorem of calculus we get

$$F(X_t^n) - F(X_0^n) := \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} F(X_s^n)\,\mathrm{d}s = \int_0^t F'(X_s^n) \frac{\mathrm{d}}{\mathrm{d}s} X_s^n\,\mathrm{d}s = \int_0^t f(X_s^n)\,\mathrm{d}X_s^n, \quad t \in [0,T].$$

(ii) Using (i), we obtain the continuous process

$$\lim_{n\to\infty} \int_0^t f(X_s^n)\,\mathrm{d}X_s^n = \lim_{n\to\infty} F(X_t^n) - F(X_0^n) = F(X_t) - F(X_0),$$

$t \in [0,T]$, where we applied that $\lim_{n\to\infty} \sup_{t\in[0,T]} |X_t^n - X_t| = 0$ in probability. Hence, the limit exists and does not depend on the approximating sequence $(X^n)_{n\in\mathbb{N}}$ as the right-hand side does not depend on it.

(iii) As $f \in C^1$, it follows that $F \in C^2$. Therefore, Itô's formula and (ii) give

$$\int_0^t f(X_s) \circ \mathrm{d}X_s := \lim_{n\to\infty} \int_0^t f(X_s^n)\,\mathrm{d}X_s^n = F(X_t) - F(X_0) = \int_0^t f(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t f'(X_s)\,\mathrm{d}\langle X\rangle_s,$$

for $t \in [0,T]$.

Exercise 5.3

(i) Let $(\Pi_n)_{n\in\mathbb{N}}$ be a zero-sequence of partitions and $t \in [0,T]$. Then

$$\begin{aligned}
\langle X, Y\rangle_t &= \frac{1}{4}(\langle X+Y\rangle_t - \langle X-Y\rangle_t) \\
&= \frac{1}{4} \lim_{n\to\infty} \sum_{J\in\Pi_n} \Big(\big(\Delta_{J\cap[0,t]}(X+Y)\big)^2 - \big(\Delta_{J\cap[0,t]}(X-Y)\big)^2 \Big) \\
&= \lim_{n\to\infty} \sum_{J\in\Pi_n} \frac{1}{4} \Big(\big(\Delta_{J\cap[0,t]}X + \Delta_{J\cap[0,t]}Y\big)^2 - \big(\Delta_{J\cap[0,t]}X - \Delta_{J\cap[0,t]}Y\big)^2 \Big) \\
&= \lim_{n\to\infty} \sum_{J\in\Pi_n} \big(\Delta_{J\cap[0,t]}X\big)\big(\Delta_{J\cap[0,t]}Y\big)
\end{aligned}$$

since $(*)$ $\frac{1}{4}((a+b)^2 - (a-b)^2) = ab$.

(ii) Itô's formula yields for $t \in [0,T]$ that

$$\begin{aligned}
f(X_t) &= f(0) + \int_0^t f'(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_s)\,\mathrm{d}\langle X\rangle_s \\
&= f(0) + \int_0^t f'(X_s)a(\cdot,s)\,\mathrm{d}s + \int_0^t f'(X_s)b(\cdot,s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(X_s)\,\mathrm{d}\langle X\rangle_s \\
&= f(0) + \int_0^t f'(X_s)a(\cdot,s)\,\mathrm{d}s + \int_0^t f'(X_s)b(\cdot,s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(X_s)b^2(\cdot,s)\,\mathrm{d}s \\
&= f(0) + \int_0^t \left(f'(X_s)a(\cdot,s) + \frac{1}{2}f''(X_s)b^2(\cdot,s) \right) \mathrm{d}s + \int_0^t f'(X_s)b(\cdot,s)\,\mathrm{d}B_s
\end{aligned}$$

and, similarly,

$$\begin{aligned}
g(Y_t) &= g(0) + \int_0^t g'(Y_s)\,\mathrm{d}Y_s + \frac{1}{2}\int_0^t g''(Y_s)\,\mathrm{d}\langle Y\rangle_s \\
&= g(0) + \int_0^t \left(g'(Y_s)\alpha(\cdot,s) + \frac{1}{2}g''(Y_s)\beta^2(\cdot,s) \right) \mathrm{d}s + \int_0^t g'(Y_s)\beta(\cdot,s)\,\mathrm{d}B_s.
\end{aligned}$$

$f(X)$ and $g(Y)$ are again two Itô processes with

$$\langle f(X), g(Y)\rangle_t = \frac{1}{4}\left(\langle f(X)+g(Y)\rangle_t - \langle f(X)-g(Y)\rangle_t\right).$$

First, we notice that

$$\begin{aligned} f(X)+g(Y) = f(0)+g(0) + \int_0^\cdot \underbrace{\left(f'(X_s)a(\cdot,s) + g'(Y_s)\alpha(\cdot,s) + \frac{1}{2}f''(X_s)b^2(\cdot,s) + \frac{1}{2}g''(Y_s)\beta^2(\cdot,s)\right)}_{=:\tilde{a}(\cdot,s)} \mathrm{d}s \\ + \int_0^\cdot \underbrace{\left(f'(X_s)b(\cdot,s) + g'(Y_s)\beta(\cdot,s)\right)}_{=:\tilde{b}(\cdot,s)} \mathrm{d}B_s \end{aligned}$$

is an Itô process with ($\mathbb{P}$-a.s.) representation

$$(f(X)+g(Y))_t = f(0)+g(0) + \int_0^t \tilde{a}(\cdot,s)\,\mathrm{d}s + \int_0^t \tilde{b}(\cdot,s)\,\mathrm{d}B_s, \quad t\in[0,T].$$

Hence,

$$\langle f(X)+g(Y)\rangle = \int_0^\cdot \left(f'(X_s)b(\cdot,s) + g'(Y_s)\beta(\cdot,s)\right)^2 \mathrm{d}s.$$

Analogously, we get

$$\langle f(X)-g(Y)\rangle = \int_0^\cdot \left(f'(X_s)b(\cdot,s) - g'(Y_s)\beta(\cdot,s)\right)^2 \mathrm{d}s,$$

which implies

$$\begin{aligned} \langle f(X), g(Y)\rangle_t &= \frac{1}{4}\left(\int_0^\cdot \left(f'(X_s)b(\cdot,s) + g'(Y_s)\beta(\cdot,s)\right)^2 \mathrm{d}s - \int_0^\cdot \left(f'(X_s)b(\cdot,s) - g'(Y_s)\beta(\cdot,s)\right)^2 \mathrm{d}s\right) \\ &= \int_0^\cdot \frac{1}{4}\left(\left(f'(X_s)b(\cdot,s) + g'(Y_s)\beta(\cdot,s)\right)^2 - \left(f'(X_s)b(\cdot,s) - g'(Y_s)\beta(\cdot,s)\right)^2\right) \mathrm{d}s \\ &\overset{(*)}{=} \int_0^\cdot f'(X_s)g'(Y_s)\underbrace{b(\cdot,s)\beta(\cdot,s)}_{=\frac{\partial}{\partial s}\langle X,Y\rangle_s} \mathrm{d}s \\ &= \int_0^\cdot f'(X_s)g'(Y_s)\,\mathrm{d}\langle X,Y\rangle_s. \end{aligned}$$

Here, we used that

$$\begin{aligned} \langle X,Y\rangle &= \frac{1}{4}(\langle X+Y\rangle - \langle X-Y\rangle) \\ &= \frac{1}{4}\left(\int_0^\cdot (b(\cdot,s)+\beta(\cdot,s))^2\,\mathrm{d}s - \int_0^\cdot (b(\cdot,s)-\beta(\cdot,s))^2\,\mathrm{d}s\right) \\ &= \int_0^\cdot \frac{1}{4}\left((b(\cdot,s)+\beta(\cdot,s))^2 - (b(\cdot,s)-\beta(\cdot,s))^2\right) \mathrm{d}s \\ &\overset{(*)}{=} \int_0^\cdot b(\cdot,s)\beta(\cdot,s)\,\mathrm{d}s. \end{aligned}$$