Exercise 4.1

(i) Assume first that f is a step function, i.e.,

$$f(t) = \sum_{i=0}^{n-1} c_i \mathbb{1}_{[t_i,t_{i+1})}(t), \quad t \in [a,b],$$

where c_i, $i = 0,\dots,n$ are constants and $a = t_0 < t_1 < \ldots < t_n = b$ is a partition of $[a,b]$. It follows

$$\mathcal{J}_{a,b} = \int_a^b f(s)\,\mathrm{d}W(s) = \sum_{i=0}^{n-1} c_i(W_{t_{i+1}} - W_{t_i}).$$

Since the increments $W_{t_{i+1}} - W_{t_i}$ are independent and normally distributed, $i = 0,\dots,n-1$, we obtain that $\mathcal{J}_{a,b}$ is normally distributed with

$$\mathbb{E}[\mathcal{J}_{a,b}] = \sum_{i=0}^{n-1} c_i \mathbb{E}[W_{t_{i+1}} - W_{t_i}] = 0$$

and

$$\mathbb{V}\mathrm{ar}(\mathcal{J}_{a,b}) = \sum_{i=0}^{n-1} c_i^2\, \mathbb{V}\mathrm{ar}(W_{t_{i+1}} - W_{t_i}) = \sum_{i=0}^{n-1} c_i^2 (t_{i+1} - t_i) = \int_a^b f^2(s)\,\mathrm{d}s.$$

Here, we used in the first step Bienaymé.
Now, consider a general $f \in L^2([0,T],\mathbb{R})$. Then there exists a sequence of step functions $(f_n)_{n\in\mathbb{N}}$ such that $f_n \to f$ in $L^2([0,T],\mathbb{R})$. Applying Itô's isometry we find

$$\int_a^b f_n(s)\,\mathrm{d}W(s) \to \int_a^b f(s)\,\mathrm{d}W(s) =: \mathcal{J}_{a,b}$$

in $L^2(\mathbb{P})$. By the given lemma and since L^2-convergence implies convergence in distribution, $\mathcal{J}_{a,b}$ is normally distributed with $\mathbb{E}[\mathcal{J}_{a,b}] = 0$ and the following variance

$$\mathbb{V}\mathrm{ar}(\mathcal{J}_{a,b}) = \lim_{n\to\infty} \mathbb{V}\mathrm{ar}\left(\int_a^b f_n(s)\,\mathrm{d}W(s)\right) = \lim_{n\to\infty} \int_a^b f_n^2(s)\,\mathrm{d}s = \int_a^b f^2(s)\,\mathrm{d}s.$$

(ii) Let $0 = t_0 < t_1 < \ldots < t_n = T$ be a partition containing the points a, b, c, d. Assume

$$f(t) = \sum_{i=0}^{n-1} c_i \mathbb{1}_{[t_i,t_{i+1})}(t), \quad t \in [a,b],$$

is a step function (similar to the previous step). Then

$$\mathcal{J}_{a,b} = \int_a^b f(s)\,\mathrm{d}W(s) = \sum_{i\in I_1} c_i(W_{t_{i+1}} - W_{t_i})$$

$$\mathcal{J}_{c,d} = \int_c^d f(s)\,\mathrm{d}W(s) = \sum_{i\in I_2} c_i(W_{t_{i+1}} - W_{t_i}),$$

where I_1, I_2 are suitable sets such that $I_1 \cap I_2$ contains at most one point. Since the increments of a Brownian motion are independent and normally distributed, we see that $\mathcal{J}_{a,b}, \mathcal{J}_{c,d}$ are independent and normally distributed. It follows also that for all $\alpha, \beta \in \mathbb{R}$, $\alpha\mathcal{J}_{a,b} + \beta\mathcal{J}_{c,d}$ is normally distributed. Consequently, $(\mathcal{J}_{a,b}, \mathcal{J}_{c,d})$ is a Gaussian vector.
Suppose $f \in L^2([0,T],\mathbb{R})$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of step functions such that $f_n \to f$ in $L^2([0,T],\mathbb{R})$. Similar to the proof of the first claim we get that

$$\alpha \int\limits_a^b f_n(s)\,\mathrm{d}W(s) + \beta \int\limits_c^d f_n(s)\,\mathrm{d}W(s) \to \alpha \int\limits_a^b f(s)\,\mathrm{d}W(s) + \beta \int\limits_c^d f(s)\,\mathrm{d}W(s) =: \alpha\mathcal{J}_{a,b} + \beta\mathcal{J}_{c,d}.$$

in $L^2(\mathbb{P})$. By the given lemma $\alpha\mathcal{J}_{a,b} + \beta\mathcal{J}_{c,d}$ is again normally distributed. Hence, $(\mathcal{J}_{a,b}, \mathcal{J}_{c,d})$ is a Gaussian random vector. Consequently, in order to show independence it is enough to show that $\mathbb{C}\mathrm{ov}(\mathcal{J}_{a,b}, \mathcal{J}_{c,d}) = 0$. We observe

$$\mathbb{C}\mathrm{ov}(\mathcal{J}_{a,b}, \mathcal{J}_{c,d}) = \frac{1}{2}(\mathbb{V}\mathrm{ar}(\mathcal{J}_{a,b} + \mathcal{J}_{c,d}) - \mathbb{V}\mathrm{ar}(\mathcal{J}_{a,b}) - \mathbb{V}\mathrm{ar}(\mathcal{J}_{c,d}))$$

and therefore, if we apply the given lemma, we obtain

$$\begin{aligned}\mathbb{C}\mathrm{ov}(\mathcal{J}_{a,b}, \mathcal{J}_{c,d}) &= \lim_{n\to\infty} \frac{1}{2}\Bigg(\mathbb{V}\mathrm{ar}\Bigg(\int\limits_a^b f_n(s)\,\mathrm{d}W(s) + \int\limits_c^d f_n(s)\,\mathrm{d}W(s)\Bigg) - \\ &\qquad \mathbb{V}\mathrm{ar}\Bigg(\int\limits_a^b f_n(s)\,\mathrm{d}W(s)\Bigg) - \mathbb{V}\mathrm{ar}\Bigg(\int\limits_c^d f_n(s)\,\mathrm{d}W(s)\Bigg)\Bigg) \\ &= \lim_{n\to\infty} \mathbb{C}\mathrm{ov}\Bigg(\int\limits_a^b f_n(s)\,\mathrm{d}W(s), \int\limits_c^d f_n(s)\,\mathrm{d}W(s)\Bigg) = 0\end{aligned}$$

Exercise 4.2 (*Prop.3.10*)

First, consider $f \in \mathscr{H}_0^2$ and let X denote the integral process of f. We show $X\mathbb{1}_{[0,\nu]} = Y\mathbb{1}_{[0,\nu]}$ where Y denotes the integral process of $f\mathbb{1}_{[0,\nu]}$. Note that by linearity of the integral it suffices to consider

$$f = a\mathbb{1}_{(r,s]} \text{ for } 0 \le r < s \le T \text{ and } a \text{ an } \mathcal{F}_r\text{-measurable r.v. with } \mathbb{E}[a^2] < \infty.$$

We discretize the stopping time ν as follows:

$$\begin{aligned} s_{i,n} &:= r + (s-r)\frac{i}{2^n}, \quad i = 0, 1, \dots, 2^n, \\ \nu^n &:= \sum_{i=0}^{2^n-1} s_{i+1,n}\mathbb{1}_{(s_{i,n}, s_{i+1,n}]}(\nu). \end{aligned}$$

Then,

$$\begin{aligned} f\mathbb{1}_{[0,\nu^n]} &= f - f\mathbb{1}_{[\nu^n,T]} \\ &= f - f\sum_{i=0}^{2^n-1} \mathbb{1}_{(s_{i,n},s_{i+1,n}]}(\nu)\mathbb{1}_{(s_{i+1,n},T]} \quad \in \mathscr{H}_0^2 \end{aligned}$$

and

$$Y_t^n := \int_0^t f(\cdot, s)\mathbb{1}_{[0,\nu^n]}(u)\mathrm{d}B_u = a(B_{s\wedge\nu^n\wedge t} - B_{r\wedge\nu^n\wedge t}).$$

Since B is continuous, it follows $Y_t = \lim_{n\to\infty} Y_t^n = a(B_{s\wedge\nu\wedge t} - B_{r\wedge\nu\wedge t})$. On the other hand, it holds $X_t = a(B_{s\wedge t} - B_{r\wedge t})$ which implies

$$X\mathbb{1}_{[0,\nu]} = Y\mathbb{1}_{[0,\nu]}. \tag{1}$$

Now, let $f \in \mathscr{H}^2$ and $(f_n)_{n\in\mathbb{N}} \subset \mathscr{H}_0^2$ such that $\|f_n - f\|_{\mathscr{H}^2} \to 0$ as $n \to \infty$ with $f_n \mathbb{1}_{[0,\nu]} = 0$, $n \in \mathbb{N}$. If X^n denotes the integral process of f_n and Y^n the integral process of $f_n \mathbb{1}_{[0,\nu]}$, it then holds

$$X\mathbb{1}_{[0,\nu]} = \lim_{n\to\infty} X^n \mathbb{1}_{[0,\nu]} \overset{(1)}{=} \lim_{n\to\infty} Y^n \mathbb{1}_{[0,\nu]} = 0. \tag{2}$$

This proves the first claim.

Further, let $f, g \in \mathscr{H}^2$ with $f\mathbb{1}_{[0,\nu]} = g\mathbb{1}_{[0,\nu]}$ and note that $f - g \in \mathscr{H}^2$ with $(f-g)\mathbb{1}_{[0,\nu]} = 0$. We apply (2) to $f - g$ and conclude

$$\int_0^{\cdot} f \mathrm{d}B_s \mathbb{1}_{[0,\nu]} = \int_0^{\cdot} g \mathrm{d}B_s \mathbb{1}_{[0,\nu]},$$

i.e., the integral processes coincide on $\mathbb{1}_{[0,\nu]}$.

Exercise 4.3

We first note that the value at time t of the given processes can be written as the value of (C^2- or) $C^{1,2}$-functions at the point (t, W_t). So, we can apply Itô's formula.

(i) Let $f^{(1)} : x \mapsto x^2$. Then, $X_t^{(1)} = f^{(1)}(W_t)$. We compute

$$\frac{\partial}{\partial x} f^{(1)}(x) = 2x, \qquad \frac{\partial^2}{\partial x^2} f^{(1)}(x) = 2.$$

Itô's formula yields

$$\begin{aligned} X_t^{(1)} &= X_0^{(1)} + \int_0^t \frac{\partial}{\partial x} f^{(1)}(W_s) dW_s + \frac{1}{2}\int_0^t \frac{\partial}{\partial x} f^{(1)}(W_s) ds \\ &= 2\int_0^t W_s dW_s + \int_0^t ds \\ &= 2\int_0^t W_s dW_s + t \end{aligned}$$

$\mathbb{P}$-a.s.

(ii) Let $f^{(2)} : (t, x) \mapsto t^2 x^3$. Then, $X_t^{(2)} = f^{(2)}(t, W_t)$. Partial differentiation gives

$$\frac{\partial}{\partial t} f^{(2)}(t,x) = 2tx^3, \qquad \frac{\partial}{\partial x} f^{(2)}(x) = 3t^2x^2, \qquad \frac{\partial^2}{\partial x^2} f^{(2)}(t,x) = 6t^2 x.$$

Itô's formula yields

$$\begin{aligned} X_t^{(2)} &= t^2 W_t^3 \\ &= X_0^{(2)} + \int_0^t \frac{\partial}{\partial t} f^{(2)}(s, W_s) ds + \int_0^t \frac{\partial}{\partial x} f^{(2)}(s, W_s) dW_s + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2} f^{(2)}(s, W_s) ds \\ &= \int_0^t \left(2sW_s^3 + 3s^2 W_s\right) ds + 3\int_0^t s^2 W_s^2 dW_s \end{aligned}$$

$\mathbb{P}$-a.s.

(iii) Let $f^{(3)} : (t, x) \mapsto \exp(mt + \sigma x)$. Then, $X_t^{(3)} = f^{(3)}(t, W_t)$. We compute

$$\begin{aligned} &\frac{\partial}{\partial t} f^{(3)}(t,x) = m \exp(mt + \sigma x), \qquad \frac{\partial}{\partial x} f^{(3)}(t,x) = \sigma \exp(mt + \sigma x), \\ &\frac{\partial^2}{\partial x^2} f^{(3)}(t,x) = \sigma^2 \exp(mt + \sigma x). \end{aligned}$$

Itô's formula yields

$$\begin{aligned}
X_t^{(3)} &= \exp\left(mt + \sigma W_t\right) \\
&= X_0^{(3)} + \int_0^t \frac{\partial}{\partial t} f^{(3)}(s, W_s) ds + \int_0^t \frac{\partial}{\partial x} f^{(3)}(s, W_s) dW_s + \frac{1}{2} \int_0^t \frac{\partial^2}{\partial x^2} f^{(3)}(s, W_s) ds \\
&= 1 + \int_0^t \left(m + \frac{1}{2}\sigma^2\right) \exp\left(ms + \sigma W_s\right) ds + \int_0^t \sigma \exp\left(ms + \sigma W_s\right) dW_s
\end{aligned}$$

$\mathbb{P}$-a.s. We can also rewrite the last equality as

$$X_t^{(3)} - X_0^{(3)} = \int_0^t \left(m + \frac{1}{2}\sigma^2\right) X_s^{(3)} ds + \int_0^t \sigma X_s^{(3)} dW_s$$

$\mathbb{P}$-a.s.

(iv) Let $f^{(4)} : (t, x) \mapsto \cos(t + x)$. Then, $X_t^{(4)} = f^{(4)}(t, W_t)$. We compute

$$\frac{\partial}{\partial t} f^{(4)}(t, x) = -\sin(t + x), \qquad \frac{\partial}{\partial x} f^{(4)}(t, x) = -\sin(t + x), \qquad \frac{\partial^2}{\partial x^2} f^{(4)}(t, x) = -\cos(t + x).$$

Itô's formula yields

$$\begin{aligned}
X_t^{(4)} &= \cos\left(t + W_t\right) \\
&= X_0^{(4)} + \int_0^t \frac{\partial}{\partial t} f^{(4)}(s, W_s) ds + \int_0^t \frac{\partial}{\partial x} f^{(4)}(s, W_s) dW_s + \frac{1}{2} \int_0^t \frac{\partial^2}{\partial x^2} f^{(4)}(s, W_s) ds \\
&= 1 - \int_0^t \left(\sin\left(s + W_s\right) + \frac{1}{2}\cos\left(s + W_s\right)\right) ds - \int_0^t \sin\left(s + W_s\right) dW_s
\end{aligned}$$

$\mathbb{P}$-a.s.

(v) Let $f^{(5)} : (t, x) \mapsto \log(2 + \cos(x - t))$. Then, $X_t^{(5)} = f^{(5)}(t, W_t)$. We compute

$$\begin{aligned}
\frac{\partial}{\partial t} f^{(5)}(t, x) &= \frac{\sin(x - t)}{2 + \cos(x - t)}, \\
\frac{\partial}{\partial x} f^{(5)}(t, x) &= -\frac{\sin(x - t)}{2 + \cos(x - t)}, \\
\frac{\partial^2}{\partial x^2} f^{(5)}(t, x) &= -\frac{\cos(x - t)\left(2 + \cos(x - t)\right) + \sin(x - t)^2}{\left(2 + \cos(x - t)\right)^2} = -\frac{1 + 2\cos(x - t)}{\left(2 + \cos(x - t)\right)^2}.
\end{aligned}$$

Itô's formula yields

$$\begin{aligned}
X_t^{(5)} &= \log\left(2 + \cos\left(W_t - t\right)\right) \\
&= X_0^{(5)} + \int_0^t \frac{\partial}{\partial t} f^{(5)}(s, W_s) ds + \int_0^t \frac{\partial}{\partial x} f^{(5)}(s, W_s) dW_s + \frac{1}{2} \int_0^t \frac{\partial^2}{\partial x^2} f^{(5)}(s, W_s) ds \\
&= \log(3) + \int_0^t \left(\frac{\sin\left(W_s - s\right)}{2 + \cos\left(W_s - s\right)} - \frac{1 + 2\cos\left(W_s - s\right)}{2\left(2 + \cos\left(W_s - s\right)\right)^2}\right) ds - \int_0^t \frac{\sin\left(W_s - s\right)}{2 + \cos\left(W_s - s\right)} dW_s
\end{aligned}$$

$\mathbb{P}$-a.s.