Stochastic Calculus
Solution Sheet 3

Exercise 3.1

(i) First, assume that the random variables $|X_n|$ are uniformly bounded by a finite constant c. Then, for $\varepsilon > 0$,

$$\begin{aligned}\mathbb{E}[|X_n - X|] &= \mathbb{E}[|X_n - X|\mathbb{1}_{\{|X_n - X| > \frac{\varepsilon}{2}\}}] + \mathbb{E}[|X_n - X|\mathbb{1}_{\{|X_n - X| \leq \frac{\varepsilon}{2}\}}] \\ &\leq 2c\mathbb{P}(|X_n - X| > \tfrac{\varepsilon}{2}) + \tfrac{\varepsilon}{2}.\end{aligned}$$

Here, we used that $|X| \leq c$ $\mathbb{P}$-a.s since $|X_n| \leq c$ and $(X_n)_{n\in\mathbb{N}}$ converges $\mathbb{P}$-a.s. to X. Further, recall that $\mathbb{P}$-a.s. convergence implies convergence in probability. So there exists $N = N(\varepsilon)$ such that the right-hand side is smaller than ε for all $n \geq N$. Therefore,

$$\mathbb{E}[|X_n - X|] \to 0 \text{ as } n \to \infty. \tag{1}$$

Now, for $c \in (0, \infty)$ we consider the function $\phi_c : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi_c(x) := (x \wedge c) \vee (-c) = \begin{cases} c & \text{if } x \geq c \\ x & \text{if } x \in (-c, c) \\ -c & \text{if } x \leq -c \end{cases}, \quad x \in \mathbb{R}.$$

Note that the function ϕ_c is a contraction, i.e. $|\phi_c(X_n) - \phi_c(X)| \leq |X_n - X|$, and

$$|\phi_c(X_n)| \leq c, \quad n \in \mathbb{N}.$$

With $X_n \to X$ $\mathbb{P}$-a.s., it follows that $\phi_c(X_n) \to \phi_c(X)$ $\mathbb{P}$-a.s. (continuous mapping theorem). Hence, for any c, by (1)

$$\mathbb{E}[|\phi_c(X_n) - \phi_c(X)|] \to 0 \text{ as } n \to \infty.$$

We want to show $\mathbb{E}[|X_n - X|] \to 0$.
Note that since $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable, for $\varepsilon = 1$ there exists a constant $L > 0$ such that

$$\begin{aligned}\mathbb{E}[|X_n|] &= \mathbb{E}[|X_n|\mathbb{1}_{\{|X_n| < L\}}] + \mathbb{E}[|X_n|\mathbb{1}_{\{|X_n| \geq L\}}] \\ &\leq L + \mathbb{E}[|X_n|\mathbb{1}_{\{|X_n| \geq L\}}] \\ &\leq L + 1, \quad n \in \mathbb{N}.\end{aligned}$$

So with $X_n \to X$ $\mathbb{P}$-a.s., we get

$$\mathbb{E}[|X|] \leq \liminf_{n\to\infty} \mathbb{E}[|X_n|] < \infty$$

by Fatou's lemma. It follows

$$\begin{aligned}\mathbb{E}[|X_n - X|] &\leq \mathbb{E}[|X_n - \phi_c(X_n)|] + \mathbb{E}[|\phi_c(X_n) - \phi_c(X)|] + \mathbb{E}[|\phi_c(X) - X|] \\ &\leq \mathbb{E}[|X_n|\mathbb{1}_{\{|X_n| \geq c\}}] + \mathbb{E}[|\phi_c(X_n) - \phi_c(X)|] + \mathbb{E}[|X|\mathbb{1}_{\{|X| \geq c\}}]\end{aligned}$$

Let $\varepsilon > 0$ be given. Choosing c large enough, by uniform integrability of $(X_n)_{n\in\mathbb{N}}$ and integrability of X the first and last summand on the right-hand side are smaller than $\frac{\varepsilon}{3}$ for all $n \in \mathbb{N}$. And there exists $N = N(c)$ such that the second summand on the right-hand side is smaller than $\frac{\varepsilon}{3}$ for all $n \geq N$. Consequently, $\mathbb{E}[|X_n - X|] < \varepsilon$ for $n \geq N$, thus $X_n \to X$ in L^1.

(ii) To show: For every $\varepsilon > 0$ there exists an $L > 0$ such that

$$\sup_{\substack{\mathcal{G}\subseteq\mathcal{F},\\ \mathcal{G}\,\sigma\text{-algebra}}} \mathbb{E}\left[|\mathbb{E}[X|\mathcal{G}]|\mathbb{1}_{\{|\mathbb{E}[X|\mathcal{G}]|\geq L\}}\right] < \varepsilon.$$

For $\mathcal{G} \subseteq \mathcal{F}$, $\mathcal{G}$ σ-algebra, we have that

$$\begin{aligned}
\mathbb{E}\left[|\mathbb{E}[X|\mathcal{G}]|\mathbb{1}_{\{|\mathbb{E}[X|\mathcal{G}]|\geq L\}}\right] &\overset{\text{Jensen's ineq.}}{\leq} \mathbb{E}\left[\mathbb{E}[|X|\,|\mathcal{G}]\mathbb{1}_{\{\mathbb{E}[|X|\,|\mathcal{G}]\geq L\}}\right] \\
&= \mathbb{E}\left[\mathbb{E}[|X|\,\mathbb{1}_{\{\mathbb{E}[|X|\,|\mathcal{G}]\geq L\}}|\mathcal{G}]\right] \\
&= \mathbb{E}\left[|X|\,\mathbb{1}_{\{\mathbb{E}[|X|\,|\mathcal{G}]\geq L\}}\right] \\
&= \mathbb{E}\left[|X|\,\left(\mathbb{1}_{\{|X|>\sqrt{L}\}} + \mathbb{1}_{\{|X|\leq\sqrt{L}\}}\right)\mathbb{1}_{\{\mathbb{E}[|X|\,|\mathcal{G}]\geq L\}}\right] \\
&\leq \mathbb{E}[|X|\mathbb{1}_{\{|X|>\sqrt{L}\}}] + \sqrt{L}\,\underbrace{\mathbb{P}\left(\mathbb{E}[|X|\,|\mathcal{G}]\geq L\right)}_{\leq\frac{\mathbb{E}[|X|]}{L},\text{ Markov's ineq.}} \\
&\leq \mathbb{E}[|X|\mathbb{1}_{\{|X|>\sqrt{L}\}}] + L^{-\frac{1}{2}}\mathbb{E}[|X|].
\end{aligned}$$

Let $\varepsilon > 0$ be given. Choosing L large enough, the summands on the right-hand side are each smaller than $\frac{\varepsilon}{2}$. Consequently, $\mathbb{E}\left[|\mathbb{E}[X|\mathcal{G}]|\mathbb{1}_{\{|\mathbb{E}[X|\mathcal{G}]|\geq L\}}\right] < \varepsilon$. Note that the upper bound holds uniformly over all σ-algebras $\mathcal{G} \subset \mathcal{F}$ which proves the claim.

Exercise 3.2

(i) By Exercise 2.3 (ii), we know that every continuous local martingale $(M_t)_{t\in[0,T]}$, which is bounded from below, is also a super-martingale and, in particular,

$$\mathbb{E}[M_t|\mathcal{F}_s] \leq M_s \quad \text{a.s.} \quad \text{for all } s,t \in [0,T] \text{ with } s \leq t.$$

Hence, it is sufficient to show that

$$\mathbb{E}[M_t|\mathcal{F}_s] = M_s \quad \text{a.s.} \quad \text{for all } s,t \in [0,T] \text{ with } s \leq t.$$

Let $s,t \in [0,T]$ with $s \leq t$. We observe that

$$0 \leq M_s - \mathbb{E}[M_t|\mathcal{F}_s] \quad \text{a.s.}$$

and, using the assumption $\mathbb{E}[M_t] = \mathbb{E}[M_0]$ for all $t \in [0,T]$,

$$\mathbb{E}[M_s - \mathbb{E}[M_t|\mathcal{F}_s]] = \mathbb{E}[M_s] - \mathbb{E}[M_t] = 0.$$

Hence, $M_s - \mathbb{E}[M_t|\mathcal{F}_s]$ is an a.s. non-negative random variable with mean zero. This implies

$$\mathbb{E}[M_t|\mathcal{F}_s] - M_s = 0 \quad \text{a.s.,}$$

which shows the claim.

(ii) (a) Let $(M_t)_{t\in[0,T]}$ be a continuous martingale in L^2.
We want to apply Theorem 2.10 (ii) $\Rightarrow$ (i): *If $(X_t)_{t\in[0,T]}$ is right-continuous and adapted and if $X_\tau \in L^1$ and $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$ for any bounded stopping time τ, then X is a martingale.*

By Theorem 2.20, $(M_t^2 - \langle M\rangle_t)_{t\in[0,T]}$ is a continuous local martingale. Then $(M_t^2 - \langle M\rangle_t)_{t\in[0,T]}$ is right-continuous and adapted.
Let $\mathscr{T} > 0$ and $\tau \leq \mathscr{T}$ be a bounded stopping time.
We first show that $\mathbb{E}[M_\tau^2 - \langle M\rangle_\tau] = \mathbb{E}[M_0^2 - \langle M\rangle_0] = 0$ or, equivalently, $\mathbb{E}[M_\tau^2] = \mathbb{E}[\langle M\rangle_\tau]$.

Again, by Theorem 2.20, $(M_t^2 - \langle M\rangle_t)_{t\in[0,T]}$ is a continuous local martingale. Let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence for $(M_t^2 - \langle M\rangle_t)_{t\in[0,T]}$.

Note that, since $\varphi(u) = u^2$ is convex and M is assumed to be square-integrable, M^2 is a nonnegative sub-martingale which implies

$$M^2_{\tau_n \wedge \tau} \leq \mathbb{E}[M^2_{\mathscr{T}} | \mathcal{F}_{\tau_n \wedge \tau}].$$

Hence, $(M^2_{\tau_n \wedge \tau})_{n \in \mathbb{N}}$ is uniformly integrable. Further, as $(\tau_n)_{n \in \mathbb{N}}$ is a localizing sequence for $(M_t^2 - \langle M \rangle_t)_{t \in [0,T]}$, for $t \in [0,T]$,

$$\begin{aligned}
\mathbb{E}[M^2_{\tau_n \wedge t} - \langle M \rangle_{\tau_n \wedge t}] &= \mathbb{E}[M^2_{\tau_n \wedge t} - \langle M \rangle_{\tau_n \wedge t} - (M_0^2 - \langle M \rangle_0)] \\
&= \mathbb{E}[M_0^2 - \langle M \rangle_0 - (M_0^2 - \langle M \rangle_0)] \\
&= 0
\end{aligned}$$

which implies $\mathbb{E}[M^2_{\tau_n \wedge t}] = \mathbb{E}[\langle M \rangle_{\tau_n \wedge t}]$, $n \in \mathbb{N}$. Hence, $\mathbb{E}[M^2_{\tau_n \wedge \tau}] = \mathbb{E}[\langle M \rangle_{\tau_n \wedge \tau}]$. Thus,

$$\mathbb{E}[M_\tau^2] = \lim_{n \to \infty} \mathbb{E}[M^2_{\tau_n \wedge \tau}] = \lim_{n \to \infty} \mathbb{E}[\langle M \rangle_{\tau_n \wedge \tau}] = \mathbb{E}[\langle M \rangle_\tau]$$

where we applied Exercise 3.1.(i) in the first step and monotone convergence ($\langle M \rangle$ non-decreasing) in the last step.
This also gives $\mathbb{E}[M_\tau^2 - \langle M \rangle_\tau] = 0 < \infty$, i.e. $M_\tau^2 - \langle M \rangle_\tau \in L^1$.
We can now conclude that $(M_t^2 - \langle M \rangle_t)_{t \in [0,T]}$ is a martingale.

(b) Again suppose that $(M_t)_{t \in [0,T]}$ is a continuous martingale in L^2, in particular, $\mathbb{E}[M_t^2] < \infty$ for all $t \in [0,T]$. Doob's L^p-inequality ($p = 2$) implies

$$\mathbb{E}\left[\sup_{t \in [0,T]} |M_t|^2\right] \leq 4\mathbb{E}[M_T^2] = 4\mathbb{E}[M_T^2 - \langle M \rangle_T] = 4\mathbb{E}[M_0^2 - \langle M \rangle_0] = 0,$$

where we used that $(M_t^2 - \langle M \rangle_t)_{t \in [0,T]}$ is a martingale by (a) and by assumption $M_0 = 0$ and $\langle M \rangle_t = 0$ for all $t \in [0,T]$.
This gives $M_t = 0$ for all $t \in [0,T]$, a.s.

Alternative proof: Suppose that M is a continuous martingale in L^2. By (a) $(M_t^2 - \langle M \rangle_t)_{t \in [0,T]}$ is a martingale. Since $\langle M \rangle_t = 0$ for all $t \in [0,T]$ it follows that M^2 is also a martingale. Then for all $t \in [0,T]$ it holds

$$\mathbb{E}[M_t^2] = \mathbb{E}[M_0^2] \overset{M_0=0}{=} 0.$$

This implies $M_t^2 = 0$ $\mathbb{P}$-a.s. for all $t \in [0,T]$, i.e. $M_t = 0$ $\mathbb{P}$-a.s. for all $t \in [0,T]$, consequently, since M is continuous, $\mathbb{P}(M_t = 0\, \forall t) = 1$, by Exercise 2.1.(i).

(c) Let $(M_t)_{t \in [0,T]}$ be a continuous local martingale, not necessarily in L^2. Consider now the localizing sequence $(\tau_n)_{n \in \mathbb{N}}$, defined by

$$\tau_n := \inf\{t \in [0,T] \,:\, |M_t| \geq n\} \wedge T, \; n \in \mathbb{N}.$$

Then, by Exercise 2.3.(i), $(M_{\tau_n \wedge t} - M_0)_{t \in [0,T]} \overset{M_0=0}{=} (M_{\tau_n \wedge t})_{t \in [0,T]} =: (M_t^{\tau_n})_{t \in [0,T]} =: M^{\tau_n}$ is a continuous martingale and in L^2 and we know, from Remark 2.22, that

$$\langle M^{\tau_n} \rangle_t = \langle M \rangle_{\tau_n \wedge t} = 0, \quad t \in [0,T],$$

for all $n \in \mathbb{N}$. So $(M_t^{\tau_n})_{\in [0,T]}$ satisfies the conditions to apply (a) and (b).
Hence, by monotone convergence we get

$$\mathbb{E}\left[\sup_{t \in [0,T]} |M_t|^2\right] = \lim_{n \to \infty} \mathbb{E}\left[\sup_{t \in [0,T]} |M_{\tau_n \wedge t}|^2\right] \overset{\text{(b)}}{=} 0.$$

As in (b) it follows $M_t = 0$ for all $t \in [0,T]$, a.s.

Exercise 3.3

We want to show

$$\int_0^t B_s \, \mathrm{d}B_s = \frac{1}{2}\big(B_t^2 - t\big), \quad t \in [0,T]$$

Let $T > t$. First note that $f = B \in \mathscr{H}^2([0,T])$ because

$$\mathbb{E}\Big[\int_0^T B_s(\omega)^2 \, \mathrm{d}s\Big] \overset{\text{Fubini}}{=} \int_0^T \mathbb{E}[B_s^2] \, \mathrm{d}s = \int_0^T s \, \mathrm{d}s < \infty.$$

We approximate B by

$$f_n(\omega, t) = \sum_{i=0}^{n-1} B_{t_i}(\omega) \mathbb{1}_{(t_i, t_{i+1}]}(t),$$

where $t_i = \frac{i}{n}T$, $i = 0, \dots, n$. Note that B_{t_i} is $\mathcal{F}_{t_i}$-measurable and $\mathbb{E}[B_{t_i}^2] = t_i^2 < \infty$, $i = 0, \dots, n$, hence, $f_n \in \mathscr{H}_0^2$. Then

$$\begin{aligned}
\|f - f_n\|_{\mathscr{H}^2}^2 &= \mathbb{E}\Big[\int_0^T \sum_{i=0}^{n-1} \mathbb{1}_{(t_i,t_{i+1}]}(t)(B_t - B_{t_i})^2 \, \mathrm{d}t\Big] \\
&\overset{\text{Fubini}}{=} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}[(B_t - B_{t_i})^2] \, \mathrm{d}t \\
&= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t - t_i) \, \mathrm{d}t \\
&= \frac{1}{2} \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 \\
&= \frac{T^2}{2n} \to 0, \quad n \to \infty.
\end{aligned}$$

Therefore, for $k_n := \max\{i : t_i < t\}$

$$\begin{aligned}
I(f\mathbb{1}_{[0,t]}) &= \lim_{n\to\infty} I(f_n \mathbb{1}_{[0,t]}) \\
&= \lim_{n\to\infty} \sum_{i \le k_n - 1} B_{t_i}(B_{t_{i+1}} - B_{t_i}) + B_{t_{k_n}}(B_t - B_{t_{k_n}}) \\
&= \lim_{n\to\infty} \sum_{i \le k_n - 1} \Big(\frac{1}{2}(B_{t_{i+1}}^2 - B_{t_i}^2) - \frac{1}{2}(B_{t_{i+1}} - B_{t_i})^2\Big) + B_{t_{k_n}}(B_t - B_{t_{k_n}}) \\
&= \lim_{n\to\infty} \frac{1}{2} \underbrace{B_{t_{k_n}}^2}_{\to B_t^2} + \underbrace{B_{t_{k_n}}(B_t - B_{t_{k_n}})}_{\to 0} - \frac{1}{2} \sum_{i \le k_n - 1} (B_{t_{i+1}} - B_{t_i})^2 = \frac{1}{2}B_t^2 - \frac{1}{2}t
\end{aligned}$$

since B is continuous, $B_0 = 0$ a.s. and

$$\lim_{n\to\infty} \sum_{i \le k_n - 1} (B_{t_{i+1}} - B_{t_i})^2 = t$$

by Corollary 2.23 ($\langle B \rangle_t = t$) in the lecture notes.