Stochastic Calculus
Solution Sheet 2

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Exercise 2.1

(i) Let us assume that X, Y are indistinguishable. Let $t \in [0,T]$. Then

$$\{\omega \in \Omega : X_t(\omega) = Y_t(\omega)\, \forall t \in [0,T]\} \subseteq \{\omega \in \Omega : X_t(\omega) = Y_t(\omega)\}$$

and it follows that

$$1 \geq \mathbb{P}(X_t = Y_t) \geq \mathbb{P}(X_t = Y_t\, \forall t \in [0,T]) = 1,$$

which implies $\mathbb{P}(X_t = Y_t) = 1$, i.e., by definition X, Y are modifications of each other.

Now, let us assume that X, Y are modifications of each other and w.l.o.g. right-continuous (the left-continuous case works analogously).
Write N_X and N_Y for the $\mathbb{P}$-null sets where X and Y are not right-continuous, respectively. Further, let $\{q_k, k \in \mathbb{N}\} = \mathbb{Q} \cap [0,T]$ be the set of rational points in $[0,T]$ and consider $N_{q_k} = \{\omega \in \Omega : X_{q_k}(\omega) \neq Y_{q_k}(\omega)\}$, $k \in \mathbb{N}$. Since X, Y are modifications of each other, we have $\mathbb{P}(N_{q_k}) = 0$, $k \in \mathbb{N}$. Hence, it holds $\mathbb{P}(N) = 0$ for $N := N_X \cup N_Y \cup \bigcup_{k\in\mathbb{N}} N_{q_k} \in \mathcal{F}$ as N is defined as a countable union of null sets.
Clearly, we have $\mathbb{P}(N^{\mathrm{c}}) = 1$ and we show that if $\omega \in N^{\mathrm{c}}$, then $X_t(\omega) = Y_t(\omega)$ for all $t \in [0,T]$. This then implies the result.
So let $\omega \in N^{\mathrm{c}}$, i.e., $(*)$ $\omega \in N_X^{\mathrm{c}} \cap N_Y^{\mathrm{c}}$ and $(**)$ $\omega \in \bigcap_{k\in\mathbb{N}} N_{q_k}^{\mathrm{c}}$, and let $t \in [0,T]$.
Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find a sequence $(q_k)_{k\in\mathbb{N}} \subset \mathbb{Q} \cap [0,T]$ in the set of rational points in $[0,T]$ such that $(q_k)_{k\in\mathbb{N}}$ is decreasing and converges to t. It follows

$$X_t(\omega) \overset{(*)}{=} \lim_{k\to\infty} X_{q_k}(\omega) \overset{(**)}{=} \lim_{k\to\infty} Y_{q_k}(\omega) \overset{(*)}{=} Y_t(\omega),$$

where we used the fact that X, Y are right-continuous. Consequently, X, Y are indistinguishable.

(ii) Let $\Omega = [0,T]$, $\mathcal{F} = \mathcal{B}([0,T])$ and $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$ with density with respect to the Lebesgue measure. Consider $X = (X_t)_{t\in[0,T]}$, where $X_t(\omega) = \mathbb{1}_{\{t=\omega\}}$, and $Y = (Y_t)_{t\in[0,T]}$, where $Y_t(\omega) = 0$, $t \in [0,T]$, $\omega \in \Omega$. Note that X is neither left- nor right-continuous.
Then X, Y are modifications of each other: Let $t \in [0,T]$. Then, $X_t(\omega) = 0 = Y_t(\omega)$, for $t \neq \omega$, and $X_t(\omega) = 1 \neq 0 = Y_t(\omega)$, for $t = \omega$.
Since $\mathbb{P}$ has a density, it holds $\mathbb{P}(\{t\}) = 0$, hence $\mathbb{P}(\{\omega \in \Omega : \omega \neq t\}) = 1$. So, we get

$$\begin{aligned}
\mathbb{P}(\{\omega \in \Omega : X_t(\omega) = Y_t(\omega)\}) &= \mathbb{P}(\{\omega : X_t(\omega) = Y_t(\omega),\, t \neq \omega\}) + \mathbb{P}(\{\omega : X_t(\omega) = Y_t(\omega),\, t = \omega\}) \\
&= \mathbb{P}(\{\omega : t \neq \omega\}) + \mathbb{P}(\emptyset) \\
&= 1 \quad \forall t \in [0,T].
\end{aligned}$$

On the other hand,

$$\mathbb{P}(X_t = Y_t\, \forall t \in [0,T]) = 1 - \mathbb{P}(\exists t \in [0,T] : X_t \neq Y_t) = 1 - \mathbb{P}(\Omega) = 0,$$

so X, Y are not indistinguishable.

Exercise 2.2 (*Lemma 2.14.*)

(i) W.l.o.g. we assume $M_0(\omega) = 0\ \forall \omega \in \Omega$. Otherwise, consider $(\tilde{M}_t)_{t\in[0,T]}$ where $\tilde{M}_t = M_t - M_0$.

(ii) Let $0 = t_0 \le \dots \le t_n = t$, $t \in [0,T]$, $n \in \mathbb{N}$. Note that, since M is a martingale,

$$\mathbb{E}\left[M_{t_i} M_{t_{i-1}}\right] = \mathbb{E}\left[\mathbb{E}\left[M_{t_i} M_{t_{i-1}} | \mathcal{F}_{t_{i-1}}\right]\right] = \mathbb{E}\left[\mathbb{E}\left[M_{t_i} | \mathcal{F}_{t_i-1}\right] M_{t_{i-1}}\right] = \mathbb{E}\left[M_{t_{i-1}}^2\right]. \tag{1}$$

Applying a telescope sum argument and since $M_0 = 0$ a.s., we have that

$$\begin{aligned}
\mathbb{E}\left[M_t^2\right] = \mathbb{E}\left[M_t^2 - M_0^2\right] = \sum_{i=1}^n \mathbb{E}\left[M_{t_i}^2 - M_{t_{i-1}}^2\right] &= \sum_{i=1}^n \mathbb{E}\left[M_{t_i}^2 - 2M_{t_{i-1}}^2 + M_{t_{i-1}}^2\right] \\
&\overset{(1)}{=} \sum_{i=1}^n \mathbb{E}\left[(M_{t_i} - M_{t_{i-1}})^2\right] \\
&= \mathbb{E}\left[\sum_{i=1}^n (M_{t_i} - M_{t_{i-1}})^2\right].
\end{aligned}$$

(iii) Let $(\Pi_n)_{n\in\mathbb{N}}$ be a sequence of partitions such that $|\Pi_n| \to 0$ as $n \to \infty$. Let $t \in T$. By abuse of notation we write $\sum_{t_i \in \Pi_n}$ or $\sup_{t_i \in \Pi_n}$, where the sum or the supremum, respectively, is taken over $0 = t_0 \le \ldots \le t_n = t$ for $(t_{i-1}, t_i] \in \Pi_n$, $i = 1, \ldots, n$. It holds that

$$\begin{aligned}
\mathbb{E}\left[M_t^2\right] &= \lim_{n\to\infty} \mathbb{E}\left[\sum_{t_i \in \Pi_n} (M_{t_i} - M_{t_{i-1}})^2\right] \\
&\le \lim_{n\to\infty} \mathbb{E}\left[\sup_{t_i \in \Pi_n} \underbrace{|M_{t_i} - M_{t_{i-1}}|}_{\substack{\le 2C' \\ \text{by ass.}}} \underbrace{\sum_{t_i \in \Pi_n} |M_{t_i} - M_{t_{i-1}}|}_{\le |M|_t \le C \text{ by ass.}}\right] \\
&\overset{\text{dom.conv.}}{\le} \mathbb{E}\left[\underbrace{\lim_{n\to\infty} \sup_{t_i \in \Pi_n} |M_{t_i} - M_{t_{i-1}}|}_{\to 0 \text{ by cont. of } M, |\Pi_n| \to 0} \cdot C\right] \\
&= 0.
\end{aligned}$$

This implies that $M_t = 0$ a.s. for all $t \in [0,T]$. Consequently, by Exercise 2.1.(i) and since M is continuous,

$$\mathbb{P}(\{\omega \in \Omega : M_t(\omega) = 0 \,\forall t \in [0,T]\}) = 1.$$

(iv) Note that $(M_t)_{t\in[0,T]}$ and $(|M|_t)_{t\in[0,T]}$ are continuous and adapted stochastic processes. By Exercise 1.2. τ_n is an $(\mathcal{F}_t)$-stopping time, $n \in \mathbb{N}$.
Hence, by optional stopping (see Theorem 2.11.) $(M_{\tau_n \wedge t})_{t\in[0,T]}$ is a martingale and satisfies the conditions in (ii) and (iii). Additionally, $\tau_n \to \infty$, $\tau_n \wedge t \to t$ and since M is continuous, we have $M_{\tau_n \wedge t} \to M_t$ as $n \to \infty$. It then holds

$$\begin{aligned}
\mathbb{E}\left[M_t^2\right] &= \mathbb{E}\left[\lim_{n\to\infty} M_{\tau_n \wedge t}^2\right] \\
&\overset{\text{Fatou}}{\le} \liminf_{n\to\infty} \mathbb{E}\left[M_{\tau_n \wedge t}^2\right] \\
&\overset{\text{(iii)}}{=} 0.
\end{aligned}$$

This again implies that $M_t = 0$ a.s. for all $t \in [0,T]$. As above we get

$$\mathbb{P}(\{\omega \in \Omega : M_t(\omega) = 0 \,\forall t \in [0,T]\}) = 1.$$

Exercise 2.3

Recall the definition of a local martingale (Definition 2.16.)

An $(\mathcal{F}_t)$-adapted process $(X_t)_{t\in[0,T]}$ is called **local martingale** if there is an increasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of $(\mathcal{F}_t)$-stopping times with $\tau_n \uparrow T$ $\mathbb{P}$-a.s. and $(X_t^n)_{t\in[0,T]} := (X_{t\wedge\tau_n} - X_0)_{t\in[0,T]}$ is an $(\mathcal{F}_t)$-martingale for every $n \in \mathbb{N}$. The sequence $(\tau_n)_{n\in\mathbb{N}}$ is called **localizing sequence** for $(X_t)_{t\in[0,T]}$.

(i) Let $\tau_n := \inf\{t \in [0,T] : |M_t| \geq n\} \wedge T$, $n \in \mathbb{N}$. Since $(M_t)_{t\in[0,T]}$ is continuous, it follows by Exercise 1.2. that τ_n is a stopping time, $n \in \mathbb{N}$. Note that by definition $(\tau_n)_{n\in\mathbb{N}}$ is an increasing sequence and $\tau_n \uparrow T$ $\mathbb{P}$-a.s.
Now, let $(\sigma_k)_{k\in\mathbb{N}}$ be the localizing sequence for $(M_t)_{t\in[0,T]}$. Then for each $n \in \mathbb{N}$ we know that $(\sigma_k)_{k\in\mathbb{N}}$ is also a localizing sequence for $(M_t^n)_{t\in[0,T]} := (M_{t\wedge\tau_n} - M_0)_{t\in[0,T]}$. So $(M_t^n)_{t\in[0,T]}$ is a continuous local martingale and bounded. Therefore Proposition 2.18.(ii) tells us that $(M_t^n)_{t\in[0,T]}$ is a martingale. This on the other hand implies that $(\tau_n)_{n\in\mathbb{N}}$ is a localizing sequence for $(M_t)_{t\in[0,T]}$.

(ii) W.l.o.g. $M_0 = 0$. Let $M_t \geq c$, $t \in [0,T]$, and $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence for $M = (M_t)_{t\in[0,T]}$. Consider the local martingale $(\tilde{M}_t)_{t\in[0,T]}$ where $\tilde{M}_t = M_t - c \geq 0$. Clearly, if $\tilde{M}$ is a super-martingale, then M is a super-martingale. Hence, we can assume $c = 0$. Consequently, applying Fatou's lemma we find for $t \in [0,T]$

$$\mathbb{E}[|M_t|] = \mathbb{E}[M_t] = \mathbb{E}[\lim_{n\to\infty} M_t^{\tau_n}] \leq \liminf_{n\to\infty} \mathbb{E}[M_t^{\tau_n}] = \liminf_{n\to\infty} \mathbb{E}[M_0^{\tau_n}] = \mathbb{E}[M_0] < \infty$$

since $M_0 \in \mathbb{R}$. Hence M is integrable. Furthermore, for $s \leq t$ we have, applying Fatou's lemma,

$$\mathbb{E}[M_t \,|\mathcal{F}_s] = \mathbb{E}[\lim_{n\to\infty} M_t^{\tau_n} \,|\mathcal{F}_s] \leq \liminf_{n\to\infty} \mathbb{E}[M_t^{\tau_n} \,|\mathcal{F}_s] = \liminf_{n\to\infty} M_s^{\tau_n} = M_s \quad \text{a.s.}$$

Since M is also adapted (as a local martingale), it follows that M is a super-martingale.