Stochastic Calculus
Solution Sheet 1

Exercise 1.1

Recall the definition of a Brownian motion (see Definition 2.1).

A real-valued stochastic process $B = (B_t)_{t\in[0,T]}$ is called a (standard one-dimensional) **Brownian motion** if

(BM0) $B_0 = 0$ a.s.

(BM1) B has independent increments, i.e., $B_{t_0} - B_0$, $B_{t_i} - B_{t_{i-1}}$, $i = 1, \dots, n$, are independent random variables for all $n \in \mathbb{N}$ and $0 \le t_0 < t_1 < \ldots < t_n \le T$.

(BM2) The increments of B are stationary and normally distributed, i.e.,

$$B_t - B_s \sim \mathcal{N}(0, |t-s|), \quad s,t \in [0,T].$$

(BM3) B has almost surely continuous sample paths, i.e., the map $t \mapsto B_t(\omega)$ is continuous for almost all $\omega \in \Omega$.

Comment: We only check (BM0) - (BM3). $(W_t^1)_{t\in[0,T]}$, $(W_t^2)_{t\in[0,T-s]}$ and $(W_t^3)_{t\in[0,T]}$ are real-valued stochastic processes.

(i) We separately check (BM0) - (BM3).

(BM0) This is clear since $W_0^1 = -B_0 = 0$ a.s.

(BM1) Let $n \in \mathbb{N}$ and $0 \le t_0 < t_1 < \ldots < t_n \le T$. Then we have that

$$B_{t_0}, \quad B_{t_i} - B_{t_{i-1}}, \quad i = 1, \dots, n$$

are independent and so are

$$W_{t_0}^1 = -B_{t_0}, \quad W_{t_i}^1 - W_{t_{i-1}}^1 = -(B_{t_i} - B_{t_{i-1}}), \quad i = 1, \dots, n.$$

(BM2) Recall that for any real-valued random variable X it holds that $X \sim \mathcal{N}(0, \sigma^2)$ if and only if $-X \sim \mathcal{N}(0, \sigma^2)$. So with

$$B_t - B_s \sim N(0, |t-s|), \quad s,t \in [0,T]$$

we can conclude that

$$W_t^1 - W_s^1 = -(B_t - B_s) \sim N(0, |t-s|), \quad s,t \in [0,T].$$

(BM3) This is clear since B has almost surely continuous paths and $x \to -x$ is a continuous function.

(ii) We separately check (BM0) - (BM3).

(BM0) This is clear since $W_0^2 = B_{s+0} - B_s = 0$ (a.s.)

(BM1) Let $n \in \mathbb{N}$ and $0 \le t_0 < t_1 < \ldots < t_n \le T - s$. Then we have that

$$\begin{aligned} W_{t_0}^2 = B_{s+t_0} - B_s \quad \text{and} \quad W_{t_i}^2 - W_{t_{i-1}}^2 &= B_{s+t_i} - B_s - (B_{s+t_{i-1}} - B_s) \\ &= B_{s+t_i} - B_{s+t_{i-1}}, \quad i = 1, \dots, n. \end{aligned} \tag{1}$$

Put $t_i' := s + t_i$ and note that $0 \le s \le t_0' < t_1' < \ldots < t_n' \le T$. With (1) and (BM1) we then get that the increments of W^2 are independent.

(BM2) Let $t, t' \in [0, T-s]$. Again, it holds

$$W_t^2 - W_{t'}^2 = B_{s+t} - B_s - (B_{s+t'} - B_s) = B_{s+t} - B_{s+t'}. \tag{2}$$

Note that $s+t, s+t' \in [0,T]$ and $s+t-(s+t') = t-t'$. With (2) and (BM2) we then conclude that

$$W_t^2 - W_{t'}^2 \sim \mathcal{N}(0, |t-t'|).$$

(BM3) This follows directly since W^2 is simply a shift in t by s minus a random variable not depending on t. So $t \to W_t^2(\omega)$ is continuous for a.e. $\omega \in \Omega$ as a composition of continuous functions.

(iii) We separately check (BM0) - (BM3).

(BM0) This is clear since $W_0^3 = \alpha B_0 + \sqrt{1-\alpha^2} B_0' = 0$ a.s.

(BM1) Let $n \in \mathbb{N}$ and $0 \le t_0 < t_1 < \cdots < t_n \le T$. Then we have

$$W_{t_0}^3 = \alpha B_{t_0} + \sqrt{1-\alpha^2} B_{t_0}', \quad W_{t_i}^3 - W_{t_{i-1}}^3 = \alpha \left(B_{t_i} - B_{t_{i-1}}\right) + \sqrt{1-\alpha^2} \left(B_{t_i}' - B_{t_{i-1}}'\right), \\ i = 1, \dots, n,$$

which are independent since B and B' are independent by assumption and we know that

$$B_{t_0}, B_{t_1} - B_{t_0}, \dots, B_{t_n} - B_{t_{n-1}} \quad \text{and} \quad B_{t_0}', B_{t_1}' - B_{t_0}', \dots, B_{t_n}' - B_{t_{n-1}}'$$

are independent.

(BM2) Let $s, t \in [0, T]$. Again, it holds

$$W_t^3 - W_s^3 = \alpha(B_t - B_s) + \sqrt{1-\alpha^2}(B_t' - B_s'). \tag{3}$$

Since B and B' are BMs, we have that

$$X_t - X_s \sim \mathcal{N}(0, |t-s|), \quad X \in \{B, B'\}. \tag{4}$$

Recall that it holds for any real-valued random variables Y_1, Y_2: if $Y_1 \sim \mathcal{N}(0, \sigma_1^2)$ and $Y_2 \sim \mathcal{N}(0, \sigma_2^2)$ are independent, then any linear combination $a_1 Y_1 + a_2 Y_2$, $a_1, a_2 \in \mathbb{R}$, is again normally distributed, particularly

$$a_1 Y_1 + a_2 Y_2 \sim \mathcal{N}(0, a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2).$$

Using this and (3) and (4), we conclude that

$$\begin{aligned} W_t^3 - W_s^3 &\sim \mathcal{N}(0, \alpha^2 |t-s| + (1-\alpha^2)|t-s|) \\ &= \mathcal{N}(0, |t-s|). \end{aligned}$$

(BM3) This is clear since W^3 is a linear combination of a.s. continuous processes. So, $t \to W_t^3(\omega)$ is continuous for a.e. $\omega \in \Omega$ as a composition of continuous functions.

(iv) Choose $B' = \pm B$. Clearly, B and B' are not independent. In this case, we have

$$W^3 = (\alpha \pm \sqrt{1-\alpha^2}) B.$$

We now show that W^3 does not satisfy (BM2) and is consequently not a Brownian motion:
Let $s, t \in [0, T]$. We have

$$W_t^3 - W_s^3 = (\alpha \pm \sqrt{1-\alpha^2})(B_t - B_s)$$

Applying (BM2) for B we get that

$$W_t^3 - W_s^3 \sim N(0, (\alpha \pm \sqrt{1-\alpha^2})^2 |t-s|).$$

But $(\alpha \pm \sqrt{1-\alpha^2})^2 \neq 1$ for $\alpha \in (0,1)$. This concludes the example showing that if B and B' are not independent, then W^3 need not be a Brownian motion.

Exercise 1.2

Recall the definition of a stopping time (see Definition 2.10)

A random variable τ with values in $[0, \infty) \cup \{\infty\}$ is called a **stopping time** (with respect to the filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$) if

$$\{\tau \le t\} \in \mathcal{F}_t \quad \text{for any } t \in [0, \infty).$$

(i) Let $t \in [0, \infty)$. We show that $\{\tau_A \le t\} \in \mathcal{F}_t$.

$(X_t)_{t\in[0,\infty)}$ is right-continuous and A is an open set, thus we have

$$\begin{aligned}\{\tau_A < t\} &= \{\omega \in \Omega \,|\, \exists\, 0 \le s < t : X_s(\omega) \in A\}\\ &= \{\omega \in \Omega \,|\, \exists\, s \in \mathbb{Q},\, 0 \le s < t : X_s(\omega) \in A\}\\ &= \bigcup_{s\in\mathbb{Q},\, 0\le s<t} \{X_s \in A\} \in \mathcal{F}_t\end{aligned}$$

since $\{X_s \in A\} \in \mathcal{F}_s \subset \mathcal{F}_t$ for all $0 \le s < t, s \in \mathbb{Q}$.

In more detail: Let $\omega \in \{\tau_A < t\}$, i.e., $\exists\, 0 \le s < t$ such that $X_s(\omega) \in A$. A is open, so $\exists\, \varepsilon > 0$ such that $B_\varepsilon(X_s(\omega)) \subset A$. And since X is right-continuous, $\exists\, \delta > 0$ such that for $\tilde{s} \in [s, s + \delta]$ $|X_{\tilde{s}}(\omega) - X_s(\omega)| < \frac{\varepsilon}{2}$, i.e., $X_{\tilde{s}} \subset B_\varepsilon(X_s(\omega)) \subset A$. We know that $\mathbb{Q}$ is dense in $\mathbb{R}$ and $0 \le s < t$, so we find $s \in \mathbb{Q}$, $0 \le s < t$ such that $X_s(\omega) \in A$.

The filtration $(F_t)_{t\in[0,\infty)}$ is assumed to be right-continuous, so it holds

$$\{\tau_A \le t\} = \bigcap_{n\in\mathbb{N}} \{\tau_A < t + \tfrac{1}{n}\} \in \bigcap_{s<t} \mathcal{F}_s = \mathcal{F}_{t+} = \mathcal{F}_t$$

since $\{\tau_A < t + \frac{1}{n}\} \in \mathcal{F}_{t+\frac{1}{n}}$ for all $n \in \mathbb{N}$.

(ii) Let $t \in [0, \infty)$. We show that $\{\tau_A \le t\} \in \mathcal{F}_t$.

Consider the open sets $A_n = \{y \in \mathbb{R} : d(y, A) < \frac{1}{n}\}$ for $n \in \mathbb{N}$ and $d(y, A) = \inf_{x\in A} |x - y|$. Then $A \subset A_n$ and $A_{n+1} \subset A_n$.
Since A is closed, it is $A = \bigcap\limits_{n\in\mathbb{N}} A_n$. X is continuous and it follows that $\tau_{A_n} \le \tau_{A_{n+1}} \le \tau_A$, $n \in \mathbb{N}$.
Claim: $\{\tau_A \le t\} = \bigcap\limits_{n\in\mathbb{N}} \{\tau_{A_n} < t\}$.
Proof: Let $\omega \in \{\tau_A \le t\}$. This implies $\tau_{A_n}(\omega) < t$ for all $n \in \mathbb{N}$, i.e., $\{\tau_A \le t\} \subset \bigcap\limits_{n\in\mathbb{N}} \{\tau_{A_n} < t\}$.
Let $\omega \in \bigcap\limits_{n\in\mathbb{N}} \{\tau_{A_n} < t\}$. Then $\forall n \in \mathbb{N}, \exists\, s \in [0, t)$ such that $X_s(\omega) \in A_n$. Since X is continuous and $A = \bigcap_{n\in\mathbb{N}} A_n$, this gives $\bigcap\limits_{n\in\mathbb{N}} \{\tau_{A_n} < t\} \subset \{\tau_A \le t\}$.
As in (i), we get that $\{\tau_{A_n} < t\} \in \mathcal{F}_t$, $n \in \mathbb{N}$. We conclude that

$$\{\tau_A \le t\} = \bigcap_{n\in\mathbb{N}} \{\tau_{A_n} < t\} \in \mathcal{F}_t.$$

Exercise 1.3

Recall the definition of a martingale (see Definition 2.8.):

Let $(M_t)_{t\in[0,T]}$ be a real-valued $(\mathcal{F}_t)$-adapted stochastic process with $\mathbb{E}[|M_t|] < \infty$ for all $t \in [0, T]$. (M_t) is called a **martingale** if $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$ a.s., for all $s, t \in [0, T]$ with $s \le t$.

(i) The Brownian motion $(B_t)_{t\in[0,T]}$ is a martingale: Indeed, $(B_t)_{t\in[0,T]}$ is a real-valued $(\mathcal{F}_t)$-adapted stochastic process and $\mathbb{E}[|B_t|] \overset{\text{CSI}}{\le} \mathbb{E}[B_t^2]^{1/2} = \sqrt{t} < \infty$ for all $t \in [0, T]$.

Let $s, t \in [0, T]$ with $s < t$. Then $B_t - B_s$ is independent of $\mathcal{F}_s$ (Markov property, Lemma 2.6.). It follows that

$$\mathbb{E}[B_t|\mathcal{F}_s] = B_s + \mathbb{E}[B_t - B_s|\mathcal{F}_s] = B_s + \mathbb{E}[B_t - B_s] = B_s \quad \text{a.s.}$$

since $B_t - B_s \sim N(0, t-s)$.
And clearly, $E[B_t|F_s] = B_s$ a.s. for $s, t \in [0,T]$, $s = t$.

(ii) Again, $(X_t^2)_{t\in[0,T]}$ is a real-valued $(\mathcal{F}_t)$-adapted stochastic process and
$\mathbb{E}[|X_t^2|] = \mathbb{E}[|B_t^2 - t|] \overset{\Delta-\text{ineq.}}{\leq} \mathbb{E}[B_t^2] + t = 2t < \infty$ for all $t \in [0,T]$. And it holds

$$\begin{aligned}
\mathbb{E}[X_t^2|\mathcal{F}_s] = \mathbb{E}[B_t^2 - t|\mathcal{F}_s] &= \mathbb{E}[(B_t - B_s + B_s)^2|\mathcal{F}_s] - t \\
&= \mathbb{E}[(B_t - B_s)^2 + 2(B_t - B_s)B_s + B_s^2|\mathcal{F}_s] - t \\
&= \mathbb{E}[(B_t - B_s)^2] + 2\mathbb{E}[B_t - B_s|\mathcal{F}_s]B_s + B_s^2 - t \\
&= (t-s) + B_s^2 - t \\
&= B_s^2 - s \quad \text{a.s.}
\end{aligned}$$

for $s, t \in [0,T]$, $s < t$. Here, we applied properties of the conditional expectation and again the Markov property and that $B_t - B_s \sim N(0, t-s)$.
And clearly, $E[X_t^2|F_s] = X_s^2$ a.s. for $s, t \in [0,T]$, $s = t$.

(iii) For $\sigma > 0$ the *stochastic exponential* $(Z_t)_{t\in[0,T]}$ of $(\sigma B_t)_{t\in[0,T]}$, given by

$$Z_t := \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right), \quad t \in [0,T],$$

is a martingale: Indeed, $(Z_t)_{t\in[0,T]}$ is a real-valued $(\mathcal{F}_t)$-adapted stochastic process.
Let $t \in [0,T]$.
Note that $B_t \overset{d}{=} \sqrt{t}B_1$ and $B_1 \sim N(0,1)$, i.e., $(*)$ $\mathbb{E}[\exp(\alpha B_1)] = \exp(\frac{1}{2}\alpha^2)$ for $\alpha \in \mathbb{R}$. It holds

$$\begin{aligned}
\mathbb{E}[|Z_t|] &= \exp\left(-\frac{1}{2}\sigma^2 t\right)\mathbb{E}\big[\exp(\sigma B_t)\big] \\
&= \exp\left(-\frac{1}{2}\sigma^2 t\right)\mathbb{E}\big[\exp(\sigma\sqrt{t}B_1)\big] \\
&\overset{(*)}{=} \exp\left(-\frac{1}{2}\sigma^2 t\right)\exp\left(\frac{1}{2}\sigma^2 t\right) = 1 < \infty.
\end{aligned}$$

Let $s, t \in [0,T]$ with $s < t$. Again note that $B_t - B_s \overset{d}{=} \sqrt{t-s}B_1$. Then the Markov property of $(B_t)_{t\in[0,T]}$ and properties of the conditional expectation give

$$\begin{aligned}
\mathbb{E}[Z_t|\mathcal{F}_s] &= \mathbb{E}\left[\exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)\middle|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[\exp\left(\sigma B_t - \sigma B_s\right)\middle|\mathcal{F}_s\right]\exp\left(\sigma B_s - \frac{1}{2}\sigma^2 t\right) \\
&= \mathbb{E}\left[\exp\left(\sigma(B_t - B_s)\right)\right]\exp\left(\sigma B_s - \frac{1}{2}\sigma^2 t\right) \\
&= \mathbb{E}\left[\exp\left(\sigma\sqrt{t-s}B_1\right)\right]\exp\left(\sigma B_s - \frac{1}{2}\sigma^2 t\right) \\
&\overset{(*)}{=} \exp\left(\frac{1}{2}\sigma^2(t-s)\right)\exp\left(\sigma B_s - \frac{1}{2}\sigma^2 t\right) \\
&= \exp\left(\sigma B_s - \frac{1}{2}\sigma^2 s\right) \\
&= Z_s \quad \text{a.s.}
\end{aligned}$$

And clearly, $E[Z_t|F_s] = Z_s$ a.s. for $s, t \in [0,T]$, $s = t$.