Stochastic Calculus
Problem Sheet 6

Exercise 6.1

Let $\alpha, \sigma > 0$ and $x_0 \in \mathbb{R}$. Use the ansatz $X_t = a(t)\big(X_0 + \int_0^t b(s)\,\mathrm{d}B_s\big)$ for differentiable functions $a, b : [0,1] \to \mathbb{R}$ to find the solution to the following stochastic differential equations:

(i) $\mathrm{d}X_t = -\alpha X_t\,\mathrm{d}t + \sigma\,\mathrm{d}B_t, \quad t \in [0,1], \quad X_0 = x_0,$

(ii) $\mathrm{d}X_t = -\frac{X_t}{1-t}\,\mathrm{d}t + \mathrm{d}B_t, \qquad t \in [0,1), \quad X_0 = 0.$

[*Remark: You can apply Proposition 5.2 but then you loose one point per part.*]

Exercise 6.2

Let $X = (X_t)_{t\in[0,T]}$ be an Itô process with $X_0 = 0$ and absolutely continuous quadratic variation $\langle X \rangle$. Consider the stochastic differential equation (SDE)

$$Z_t = 1 + \int_0^t Z_s\,\mathrm{d}X_s, \quad t \in [0,T]. \tag{1}$$

Show that the unique solution $(Z_t)_{t\in[0,T]}$ of the SDE (1) is given by

$$Z_t = \mathcal{E}(X)_t := \exp\left(X_t - \frac{1}{2}\langle X\rangle_t\right), \quad t \in [0,T].$$

We call $\mathcal{E}(X)$ the *stochastic exponential* of X.

Exercise 6.3

For $\mu, \sigma \in C^1(\mathbb{R})$ let X be a solution to

$$X_t = x_0 + \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dB_s, \quad t \in [0,T].$$

Let $u \in C^{2,1}(\mathbb{R} \times [0,T])$ be a bounded solution to *Kolmogorov's backward equation*

$$\begin{cases} \partial_t u(x,t) = \mathcal{L}u(x,t), & x \in \mathbb{R},\, t \in (0,T] \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases}$$

where $f \in C(\mathbb{R})$ is compactly supported and $\mathcal{L}g := \frac{1}{2}\sigma^2 g_{xx} + \mu g_x$ for $g \in C^{2,1}(\mathbb{R} \times [0,T])$. Prove that

$$u(x_0,t) = \mathbb{E}^{x_0}[f(X_t)], \qquad \text{for } t \in [0,T] \text{ and any initial value } x_0 \in \mathbb{R}.$$

[*Hint: Fix $t_0 \in (0,T]$ and prove that $M = (M_t)_{t\in[0,t_0)}$, with $M_t = u(X_t, t_0 - t)$, $t \in [0,t_0]$, is a martingale using Itô's formula.*]

Please submit your solutions by Tuesday, the 19th of October, at noon (12 pm).

Programming exercise 6

Let $X = (X_t)_{t\in[0,T]}$ be a solution of the SDE

$$\mathrm{d}X_t = \mu(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t, \quad t \in [0, T], \quad X_0 = x_0$$

where $(B_t)_{t\in[0,T]}$ is a one-dimensional Brownian motion. For the equidistant grid given by $t_i = \frac{i}{n}T$, $i = 0, \dots, n$, the *Euler-Maruyama approximation* $Y = (Y_i, i \in \{0, \dots, n\})$ of X is given by $Y_0 = x_0$ and

$$Y_{i+1} = Y_i + \mu(t_i, Y_i)\frac{T}{n} + \sigma(t_i, Y_i)(B_{t_{i+1}} - B_{t_i}), \qquad i = 0, \dots, n-1,$$

Implement this approximation scheme for

(i) $\mathrm{d}X_t = X_t\,\mathrm{d}t + X_t\,\mathrm{d}B_t$, $t \in [0, T]$ with $X_0 = 1$ and

(ii) $\mathrm{d}X_t = -X_t\,\mathrm{d}t + \mathrm{d}B_t$, $t \in [0, T]$ with $X_0 = 1$.

Choose an appropriate time horizon T and step size n. For (i), also visualize the exact solution that you know from the lecture in the plot and compare it to the *Euler-Maruyama approximation.*