Stochastic Calculus
Problem Sheet 5

Exercise 5.1

(i) Let $(X_t)_{t\in[0,T]}$ be an Itô process with representations

$$X_t = X_0 + \int_0^t a(\cdot,s)\,\mathrm{d}s + \int_0^t b(\cdot,s)\,\mathrm{d}B_s = \tilde{X}_0 + \int_0^t \tilde{a}(\cdot,s)\,\mathrm{d}s + \int_0^t \tilde{b}(\cdot,s)\,\mathrm{d}B_s$$

for $t \in [0,T]$ and $X_0 = \tilde{X}_0$ $\mathbb{P}$-a.s. Prove that $a = \tilde{a}$ and $b = \tilde{b}$ $\mathbb{P}\otimes\lambda$-a.s.

(ii) Let $(X_t)_{t\in[0,T]}$ and $(Y_t)_{t\in[0,T]}$ be two Itô processes. Prove the product rule:

$$X_tY_t = X_0Y_0 + \int_0^t Y_s\,\mathrm{d}X_s + \int_0^t X_s\,\mathrm{d}Y_s + \langle X, Y\rangle_t$$

for all $t \in [0,T]$, $\mathbb{P}$-a.s.

Exercise 5.2

Let $X = (X_t)_{t\in[0,T]}$ be an Itô process and $X^n = (X^n_t)_{t\in[0,T]}$, $n \in \mathbb{N}$, be a sequence of stochastic processes such that

- $t \mapsto X^n_t(\omega)$ are C^1-functions for all $\omega \in \Omega$, $n \in \mathbb{N}$, and
- $\lim_{n\to\infty} \sup_{t\in[0,T]} |X^n_t - X_t| = 0$ in probability.

Further, let $f \in C^1(\mathbb{R})$.

(i) Show that

$$\int_0^t f(X^n_s)\,\mathrm{d}X^n_s = F(X^n_t) - F(X^n_0)$$

for $n \in \mathbb{N}$, $t \in [0,T]$, where $F(x) := \int_0^x f(y)\,\mathrm{d}y$.

(ii) Conclude that there exists a continuous process

$$\int_0^t f(X_s)\circ \mathrm{d}X_s := \lim_{n\to\infty} \int_0^t f(X^n_s)\,\mathrm{d}X^n_s, \quad t \in [0,T],$$

which does not depend on the approximating sequence $(X^n)_{n\in\mathbb{N}}$.

(iii) Show that

$$\int_0^t f(X_s)\circ \mathrm{d}X_s = \int_0^t f(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t f'(X_s)\,\mathrm{d}\langle X\rangle_s$$

for t $\in [0,T]$.

[Remark: $\int_0^t f(X_s)\circ \mathrm{d}X_s$ is called *Stratonovich integral* of $f(X)$ with respect to X.]

Exercise 5.3

Let $(X_t)_{t\in[0,T]}$ and $(Y_t)_{t\in[0,T]}$ be two Itô processes with representations

$$X_t = \int_0^t a(\cdot,s)\,\mathrm{d}s + \int_0^t b(\cdot,s)\,\mathrm{d}B_s \quad \text{and} \quad Y_t = \int_0^t \alpha(\cdot,s)\,\mathrm{d}s + \int_0^t \beta(\cdot,s)\,\mathrm{d}B_s, \quad t\in[0,T].$$

(i) Prove that for any zero-sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$ and all $t\in[0,T]$:

$$\langle X,Y\rangle_t = \lim_{n\to\infty} \sum_{J\in\Pi_n} (\Delta_{J\cap[0,t]}X)(\Delta_{J\cap[0,t]}Y) \qquad \text{in probability.}$$

(ii) Show for $f,g\in C^2(\mathbb{R})$ that

$$\langle f(X),g(Y)\rangle_t = \int_0^t f'(X_s)g'(Y_s)\,\mathrm{d}\langle X,Y\rangle_s \qquad \text{for } t\in[0,T].$$

Please submit your solutions by Tuesday, the 12th of October, at noon (12 pm).

Programming exercise 5

Doing this exercise is optional! Do not submit your solution for correction. If you found an elegant solution, please do submit it so that we can improve our sample solution and thus help all students.

From Exercise 5.3 (i) we know that the quadratic covariation of two Itô processes $(X_t)_{t\in[0,T]}$ and $(Y_t)_{t\in[0,T]}$ can be written as

$$\langle X, Y\rangle_t = \lim_{n\to\infty} \sum_{J\in\Pi_n} (\Delta_{J\cap[0,t]}X)(\Delta_{J\cap[0,t]}Y) \qquad \text{in probability,}$$

for any zero-sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$ and all $t \in [0,T]$.

(i) Let $(B_t^{(1)})_{t\in[0,T]}$ and $(B_t^{(2)})_{t\in[0,T]}$ be two independent Brownian motions, and define for $\rho \in [0,1]$:

$$\tilde{B}_t := \rho B_t^{(1)} + \sqrt{1-\rho^2} B_t^{(2)}, \quad t \in [0,T].$$

Show graphically that

$$\langle B^{(1)}, \tilde{B}\rangle_t = \rho t, \quad t \in [0,T],$$

and

$$\langle B^{(2)}, \tilde{B}\rangle_t = \sqrt{1-\rho^2}\, t, \quad t \in [0,T].$$

(ii) Consider the Itô processes

$$X_t = X_0 + \int_0^t a(\cdot,s)ds + \int_0^t b(\cdot,s)\, \mathrm{d}B_s, \quad t \in [0,T],$$

and

$$\tilde{X}_t = X_0 + \int_0^t a(\cdot,s)ds + \int_0^t \tilde{b}(\cdot,s)\, \mathrm{d}B_s, \quad t \in [0,T],$$

where $X_0 = 0$, $a(\cdot,t) = t$, $b(\cdot,t) = \sqrt{2t}$ and $\tilde{b}(\cdot,t) = \frac{3}{\sqrt{2}}t^2$, for $t \in [0,T]$.

Show graphically that

$$\langle X\rangle_t = \int_0^t b^2(\cdot,s)\, \mathrm{d}s, \quad t \in [0,T],$$

and

$$\langle X, \tilde{X}\rangle_t = \int_0^t b(\cdot,s)\tilde{b}(\cdot,s)\, \mathrm{d}s, \quad t \in [0,T].$$