HWS 2021
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Stochastic Calculus
Problem Sheet 4

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Exercise 4.1

Let $(W_t)_{t\in[0,T]}$ be a Brownian motion. Let $f \in L^2([0,T];\mathbb{R})$ be a deterministic function. For any $0 \le a < b \le T$ we set

$$\mathcal{J}_{a,b} := \int_a^b f(s)\,\mathrm{d}W_s.$$

(i) Show that $\mathcal{J}_{a,b}$ is normally distributed with $\mathbb{E}[\mathcal{J}_{a,b}] = 0$ and $\mathbb{V}\mathrm{ar}(\mathcal{J}_{a,b}) = \int_a^b f^2(s)\,\mathrm{d}s$.

[*Hint: Assume first that f is a step function and use the lemma stated below.*]

(ii) Let $0 \le a < b \le c < d \le T$. Show that the random vector $(\mathcal{J}_{a,b}, \mathcal{J}_{c,d})$ is a Gaussian vector and that the random variables $\mathcal{J}_{a,b}$ and $\mathcal{J}_{c,d}$ are independent.

[*Hint: Assume again first that f is a step function and use the lemma stated below.*]

Lemma: Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables with $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ for each $n \in \mathbb{N}$. If $(X_n)_{n\in\mathbb{N}}$ converges in distribution to a random variable X, i.e. $X_n \xrightarrow{d} X$, then the limits $\mu := \lim\limits_{n\to\infty} \mu_n$ and $\sigma^2 := \lim\limits_{n\to\infty} \sigma_n^2$ exist and $X \sim \mathcal{N}(\mu, \sigma^2)$.

Exercise 4.2

Prove Proposition 3.10.:
Let $f \in \mathscr{H}^2$ and ν be a stopping time satisfying $f\mathbb{1}_{[0,\nu]} = 0$. The integral process $X = (X_t)_{t\in[0,T]}$, with $X_t = \int_0^t f(\cdot, s) dB_s$, then fulfills $X\mathbb{1}_{[0,\nu]} = 0$. In particular, for two functions $f, g \in \mathscr{H}^2$ with $f\mathbb{1}_{[0,\nu]} = g\mathbb{1}_{[0,\nu]}$ the integral processes coincide on $[0,\nu]$.

[*Hint: Show first by discretizing ν that if $f \in \mathscr{H}_0^2$ and Y denotes the integral process of $f\mathbb{1}_{[0,\nu]}$, then $X\mathbb{1}_{[0,\nu]} = Y\mathbb{1}_{[0,\nu]}$.*]

Exercise 4.3

Let $W = (W_t)_{t\in[0,T]}$ be a Brownian motion. Use Itô's formula to write the following processes as integrals w.r.t. W and t:

(i) $X_t^{(1)} = W_t^2$,

(ii) $X_t^{(2)} = t^2 W_t^3$,

(iii) $X_t^{(3)} = \exp(mt + \sigma W_t)$,

(iv) $X_t^{(4)} = \cos(t + W_t)$,

(v) $X_t^{(5)} = \log\big(2 + \cos(W_t - t)\big)$.

Please submit your solutions by Tuesday, the 5th of October, at noon (12 pm).

Programming exercise 4

Doing this exercise is optional! Do not submit your solution for correction. If you found an elegant solution, please do submit it so that we can improve our sample solution and thus help all students.

In this exercise, we will show that stochastic integration (in contrast to Riemann integration) is not unique in terms of where to evaluate the integrand. Define therefore for general $(A_t)_{t\in[0,T]}$ and $f\colon [0,T] \to \mathbb{R}$, and for some constant $\alpha \in [0,1]$, the α-integrals

$$\int_0^t f(s)\,\mathrm{d}^\alpha A_s := \lim_{n\to\infty} \sum_{t_i^n \in \Pi^n} \Big(f(t_i^n) + \alpha\Big(f(t_{i+1}^n) - f(t_i^n)\Big)\Big) \cdot \Big(A_{t_{i+1}^n \wedge t} - A_{t_i^n \wedge t}\Big), \tag{1}$$

where $(\Pi^n)_{n\in\mathbb{N}}$ is any zero-sequence of partitions of $[0,T]$. Show numerically (by approximating both sides of the equations in one plot respectively), that:

(i) For $A_t := t$, one has

$$\int_0^t B_s\,\mathrm{d}^0 s = \int_0^t B_s\,\mathrm{d}^{\frac{1}{2}} s = \int_0^t B_s\,\mathrm{d}^1 s = 2tB_t - \int_0^t s\,\mathrm{d}B_s, \quad \forall t \in [0,T].$$

(ii) For $A_t := B_t$ $\forall t \in [0,T]$ for some Brownian motion $(B_t)_{t\in[0,T]}$, one has

(a)

$$\int_0^t B_s\,\mathrm{d}^0 B_s = \frac{1}{2}(B_t^2 - t),$$

(b)

$$\int_0^t B_s\,\mathrm{d}^{\frac{1}{2}} B_s = \frac{1}{2}B_t^2,$$

(c)

$$\int_0^t B_s\,\mathrm{d}^1 B_s = \frac{1}{2}(B_t^2 + t).$$

for $t \in [0,T]$.

Remark. *In (ii), the stochastic integral in (a) is the well-known Itô integral, while (b) and (c) are the so-called 'Stratonovich' and 'backward-Itô' integrals. If f is a stochastic process, then the integrals in (b) and (c) are not adapted anymore, in contrast to the Itô integral. This is due to the fact that you need to 'look a bit into the future' of the process f to evaluate the integrals in (b) and (c). On the other hand, calculating with the Stratonovich integral has its advantages since the chain rule of ordinary calculus holds here (instead of Itô's formula).*