Stochastic Calculus
Problem Sheet 3

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Exercise 3.1

Let X and X_n, $n \in \mathbb{N}$, be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove the following statements:

(i) If $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable and $X_n \to X$ $\mathbb{P}$-a.s., then $X_n \to X$ in L^1.

(ii) If X is integrable, then the family $\{\mathbb{E}[X|\mathcal{G}] : \mathcal{G} \subseteq \mathcal{F} \text{ is a } \sigma\text{-algebra}\}$ is uniformly integrable.

Recall that a family of random variables $(Y_i)_{i\in I}$ is called uniformly integrable if for every $\varepsilon > 0$ there exists an $L > 0$ such that

$$\sup_{i\in I} \mathbb{E}\big[|Y_i| \, \mathbb{1}_{|Y_i|\geq L}\big] \leq \varepsilon.$$

Exercise 3.2

Let $(M_t)_{t\in[0,T]}$ be a continuous local martingale with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and suppose that $T < \infty$. Prove the following statements:

(i) If $(M_t)_{t\in[0,T]}$ is bounded from below and $\mathbb{E}[M_t] = \mathbb{E}[M_0] < \infty$ for all $t \in [0,T]$, then $(M_t)_{t\in[0,T]}$ is a martingale.

(ii) If $M_0 = 0$ and $\langle M\rangle_t = 0$ for all $t \in [0,T]$, then $M_t = 0$ for all $t \in [0,T]$, $\mathbb{P}$-a.s. Proceed as follows:

a) Show that if $(M_t)_{t\in[0,T]}$ is a continuous square-integrable martingale, then $(M_t^2 - \langle M\rangle_t)_{t\in[0,T]}$ is a martingale.

b) Assume that M is a continuous square-integrable martingale and show the claim.

c) Extend the argument in b) by a localization argument to obtain the result for the general case.

Exercise 3.3

Let $(B_t)_{t\in[0,T]}$ be a Brownian motion. Show that

$$\int_0^t B_s \, \mathrm{d}B_s = \frac{1}{2}\big(B_t^2 - t\big), \quad t \in [0,T].$$

Please submit your solutions by Tuesday, the 28th of September, at noon (12 pm).

Programming exercise 3

Doing this exercise is optional! Do not submit your solution for correction. If you found an elegant solution, please do submit it so that we can improve our sample solution and thus help all students.

In this exercise, your task is to approximate pathwise Lebesgue–Stieltjes integrals of the form

$$X_t := \int_0^t f(s, B_s)\,\mathrm{d}s, \quad t \in [0, T]. \tag{1}$$

where $B = (B_t)_{t\in[0,T]}$ is a Brownian motion, and stochastic Itô integrals of the form

$$Y_t := \int_0^t g(s, B_s)\,\mathrm{d}B_s, \quad t \in [0, T]. \tag{2}$$

To do that, use an equal width discretization of $[0, T]$ with N discretization steps per time unit, and the approximation formulas

$$\hat{X}_{t_{n+1}} = \hat{X}_{t_n} + f(t_n, B_{t_n})\Delta_n, \quad n = 0, \dots, TN - 1, \tag{3}$$

and

$$\hat{Y}_{t_{n+1}} = \hat{Y}_{t_n} + g(t_n, B_{t_n})\Delta B_n, \quad n = 0, \dots, TN - 1, \tag{4}$$

where $\Delta_n := t_{n+1} - t_n$ and $\Delta B_n := B_{t_{n+1}} - B_{t_n}$. Use (3) and (4) to visualize that the following equalities hold:

(i)
$$\int_0^t B_s\,\mathrm{d}B_s = \frac{1}{2}(B_t^2 - t),$$

(ii)
$$\int_0^t B_s^2\,\mathrm{d}B_s = \frac{1}{3}B_t^3 - \int_0^t B_s\,\mathrm{d}s,$$

(iii)
$$\int_0^t s^2\,\mathrm{d}B_s = t^2 B_t - 2\int_0^t sB_s\,\mathrm{d}s.$$

Plot your approximations of the left- and right-hand sides of (i), (ii) and (iii) with $T = 5$ and $N = 5$, 100, 10000 to visualize that the approximations (3) and (4) converge against the integrals (1) and (2).

Remark. *(a) The approximation in* (3) *is a simple Riemann sum, and we could therefore also evaluate f on the right interval bound $(t_{n+1}, B_{t_{n+1}})$ here. The approximation in* (4) *though is a simple form of the so-called Euler-scheme. Evaluating g on $(t_{n+1}, B_{t_{n+1}})$ would yield a wrong convergence here.*

(b) It is easy to prove the equalities (i), (ii) and (iii) by using Itô's formula, which is the equivalent of the fundamental theorem of calculus for the stochastic integration theory, and will be content of the lecture soon.