HWS 2021
Prof. Dr. David Prömel
Anna Kwossek

Stochastic Calculus
Tutorial 1

UNIVERSITÄT MANNHEIM
Fakultät für Wirtschaftsinformatik und Wirtschaftsmathematik

Stopping times

Definition. A random variable τ with values in $[0,T]\cup\{\infty\}$ is called **stopping time** (with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$) if

$$\{\tau \le t\} \in \mathcal{F}_t \quad \text{for any } t\in[0,T].$$

Interpretation: If τ is a stopping time, one can tell up to time t (based on the information in $\mathcal{F}_t$) whether $\tau \le t$ or not.

Lemma. *Let σ, τ be stopping times. Then:*

i) $\sigma\wedge\tau$ and $\sigma\vee\tau$ are stopping times.

ii) If $\sigma,\tau\ge 0$, then $\sigma+\tau$ is a stopping time.

iii) If $s\ge 0$, then $\tau+s$ is a stopping time. However, in general, $\tau-s$ is not.

Proof. Let $t\in[0,T]$.

i) It holds $\{\sigma\vee\tau\le t\}=\{\sigma\le t\}\cup\{\tau\le t\}\in\mathcal{F}_t$. Analogously, $\{\sigma\wedge\tau\le t\}=\{\sigma\le t\}\cap\{\tau\le t\}\in\mathcal{F}_t$.

ii) Clearly, t is a stopping time. By i) it then follows that $\sigma\wedge t$ and $\tau\wedge t$ are stopping times. So, $\{\tau\wedge t\le s\}\in\mathcal{F}_s\subset\mathcal{F}_t$ for $s\le t$. And it holds $\tau\wedge t\le s$ for $s>t$. Hence $\sigma' := (\sigma\wedge t)+\mathbb{1}_{\{\tau>t\}}$ and $\tau' := (\tau\wedge t)+\mathbb{1}_{\{\sigma>t\}}$ are $\mathcal{F}_t$-measurable, and so is $\sigma'+\tau'$. Consequently, $\{\sigma+\tau\le t\} = \{\sigma'+\tau'\le t\}\in\mathcal{F}_t$.

iii) By ii) $\tau+s$ is a stopping time since s is a stopping time.
Now note that $\{\tau-s\le t\}=\{\tau\le s+t\}\in F_{s+t}$. But in general, $\mathcal{F}_{s+t}\supsetneq\mathcal{F}_t$. So $\tau-s$ need not be a stopping time.

□

Definition. Let τ be a stopping time.

$$F_\tau := \{A\in\mathcal{F} : A\cap\{\tau\le t\}\in\mathcal{F}_t \text{ for any } t\in[0,T]\}$$

is called **σ-algebra of τ-past**.

Lemma. *If σ, τ are stopping times with $\sigma\le\tau$, it holds $\mathcal{F}_\sigma\subset\mathcal{F}_\tau$.*

Proof. Let $A\in\mathcal{F}_\sigma$ and $t\in[0,T]$. Then, $A\cap\{\sigma\le t\}\in\mathcal{F}_t$ and $\{\tau\le t\}\in\mathcal{F}_t$ since τ is a stopping time. And since $\sigma\le\tau$, it follows

$$A\cap\{\tau\le t\} = (A\cap\{\sigma\le t\})\cap\{\tau\le t\} \in \mathcal{F}_t$$

which shows $A\in\mathcal{F}_\tau$. □

Hints for Exercise Sheet 2

Exercise 2.2

(ii) Apply a telescope sum argument.

(iii) Consider a sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$ such that $|\Pi_n|\to\infty$ as $n\to\infty$, use the arguments in ii) and apply dominated convergence.

(iv) We want to apply optional stopping and Fatou's lemma.

Exercise 2.3

(i) We want to apply Proposition 2.19 b).

(ii) Again, we can apply Fatou's lemma.