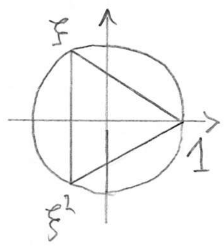
FSS 2020 LA IIb Lösungen zu Blatt 1

1 (a)

$$\xi = e^{2\pi i\, 1/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} \cdot i$$

$$\xi^2 = e^{2\pi i\, 2/3} = -\frac{1}{2} - \frac{\sqrt{3}}{2} \cdot i$$

$$|1-\xi| = \sqrt{\left(1-\left(-\tfrac{1}{2}\right)\right)^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{\frac{12}{4}} = \sqrt{3}$$



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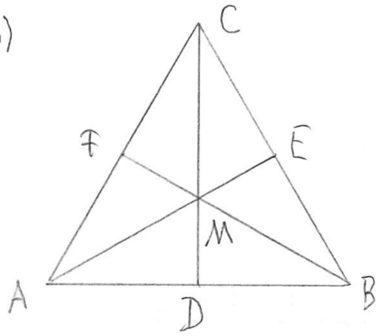
(b)



$$|AE| = \frac{|1-(-\frac{1}{2})|}{|1-\xi|} = \frac{3/2}{\sqrt{3}} = \frac{\sqrt{3}}{2},$$

$$|AM| = \frac{|1-0|}{|1-\xi|} = \frac{1}{\sqrt{3}},$$

$$|ME| = \frac{|0-(-\frac{1}{2})|}{|1-\xi|} = \frac{1}{2\sqrt{3}},$$

$$|MA| / |ME| = 2.$$

(c) $|AD| = \frac{1}{2}|AB| = \frac{1}{2} \cdot 1 = \frac{1}{2}$.

$|AE|$ mit Pythagoras: $1 = |AB|^2 = |AE|^2 + |EB|^2 = |AE|^2 + \frac{1}{4}$,

$$|AE|^2 = \frac{3}{4}, \quad |AE| = \frac{\sqrt{3}}{2}.$$

$$\text{Streckungsfaktor} = \frac{|AE|}{|AD|} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3},$$

$$|MA| = \frac{|AB|}{\text{Streckungsfaktor}} = \frac{1}{\sqrt{3}}.$$

$$|ME| = |MD| = \frac{|EB|}{\text{Streckungsfaktor}} = \frac{1/2}{\sqrt{3}} = \frac{1}{2 \cdot \sqrt{3}}.$$

(d) $|DN|$ mit Pythagoras und b):

$$1 = |AD|^2 = |AN|^2 + |DN|^2 \underset{(b)}{=} \frac{1}{3} + |DN|^2, \quad |DN| = \sqrt{\frac{2}{3}}.$$

2]

(b) $\Delta(D,K,N)$ · Streckungsfaktor ist kongruent zu $\Delta(A,M,D)$.

$$\text{Streckungsfaktor} = \frac{|DN|}{|DK|} = \frac{\sqrt{\frac{2}{3}}}{\frac{1}{2}} = \sqrt{\frac{8}{3}} .$$

$$|DM| = \frac{|DA|}{\text{Streckungsfaktor}} = \frac{1}{\sqrt{8/3}} = \sqrt{\frac{3}{8}} = R_{\text{Tetraeder}} .$$

$$|MN| = |DN| - |DM| = \sqrt{\frac{2}{3}} - \sqrt{\frac{3}{8}} = \frac{4}{\sqrt{24}} - \frac{3}{\sqrt{24}} = \frac{1}{\sqrt{24}} .$$

$$\frac{|DM|}{|MN|} = \frac{\sqrt{3/8}}{1/\sqrt{24}} = 3 .$$

$\boxed{2}$ (a) Koordinaten der 8 Ecken des Würfels:

$\left(\frac{1}{2}\cdot\varepsilon_1, \frac{1}{2}\cdot\varepsilon_2, \frac{1}{2}\cdot\varepsilon_3\right)$ mit $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\}$.

$$R_{\text{Würfel}} = \left\| \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \right\| = \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = \frac{\sqrt{3}}{2} .$$

(b) Koordinaten der 6 Ecken des Oktaeders:

$\left(\pm\frac{1}{\sqrt{2}}, 0, 0\right), \left(0, \pm\frac{1}{\sqrt{2}}, 0\right), \left(0, 0, \pm\frac{1}{\sqrt{2}}\right)$.

$$R_{\text{Oktaeder}} = \left\| \left(\frac{1}{\sqrt{2}}, 0, 0\right) \right\| = \frac{1}{\sqrt{2}} .$$

$\boxed{3}$ (a)
$$1 = \left(\cos\frac{2\pi}{5}\right)^2 + \left(\sin\frac{2\pi}{5}\right)^2 = \left(\frac{\sqrt{5}-1}{4}\right)^2 + \left(\sin\frac{2\pi}{5}\right)^2$$
$$= \frac{5-2\sqrt{5}+1}{16} + \left(\sin\frac{2\pi}{5}\right)^2, \text{ also } \left(\sin\frac{2\pi}{5}\right)^2 = \frac{5+\sqrt{5}}{8},$$
$$\sin\frac{2\pi}{5} = \sqrt{\frac{5+\sqrt{5}}{8}} .$$

$$1 = \left(\cos\frac{4\pi}{5}\right)^2 + \left(\sin\frac{4\pi}{5}\right)^2 = \left(\frac{-\sqrt{5}-1}{4}\right)^2 + \left(\sin\frac{4\pi}{5}\right)^2$$
$$= \frac{5+2\sqrt{5}+1}{16} + \left(\sin\frac{4\pi}{5}\right)^2, \text{ also } \left(\sin\frac{4\pi}{5}\right)^2 = \frac{5-\sqrt{5}}{8},$$
$$\sin\frac{4\pi}{5} = \sqrt{\frac{5-\sqrt{5}}{8}} .$$

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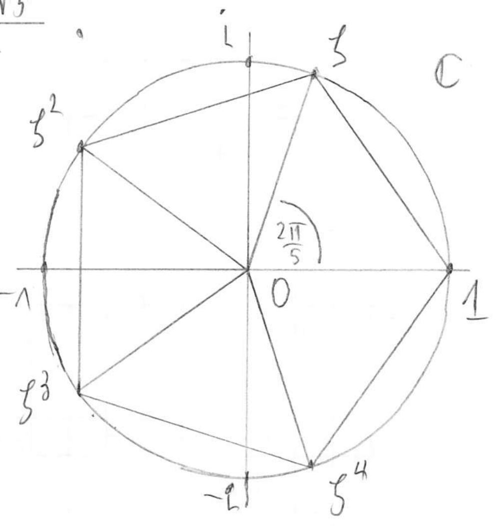
$$|1-\zeta| = |\zeta^2-\zeta^3| = 2\cdot \sin\frac{4\pi}{5} = \sqrt{\frac{5-\sqrt{5}}{2}} .$$



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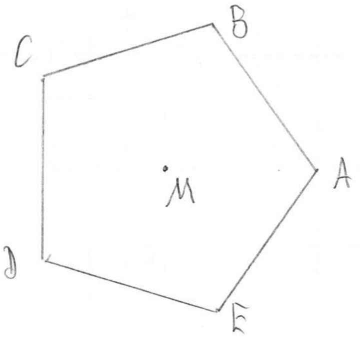
(b)



$$|MA| = \frac{1}{|1-\zeta|} = \sqrt{\frac{2}{5-\sqrt{5}}} .$$

(c) $|NP|$ mit Pythagoras und (b):

$$1 = |AP|^2 = |NP|^2 + |NA|^2 \underset{(b)}{=} |NP|^2 + \frac{2}{5-\sqrt{5}} ,$$

also $|NP|^2 = 1 - \frac{2}{5-\sqrt{5}} = \frac{3-\sqrt{5}}{5-\sqrt{5}}$, also $|NP| = \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}}$.

(d)
$$|MP| = |AP| \cdot \frac{|RP|}{|NP|} = 1 \cdot \frac{1/2}{\sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}}} = \frac{1}{2}\cdot\sqrt{\frac{5-\sqrt{5}}{3-\sqrt{5}}}$$
$$= R_{\text{Ikosaeder}} .$$

$\boxed{4}$ Aufgabe $\boxed{1}$ $\Rightarrow$ $|PK| = \frac{1}{\sqrt{3}}$, $|RK| = \frac{1}{2\cdot\sqrt{3}}$.

$|RM|$ mit Pythagoras und Aufgabe $\boxed{3}$ (d):

$$\frac{1}{4}\,\frac{5-\sqrt{5}}{3-\sqrt{5}} \underset{\boxed{3}(d)}{=} |MP|^2 = |RM|^2 + |RP|^2 = |RM|^2 + \frac{1}{4} ,$$

also $|RM|^2 = \frac{1}{4}\,\frac{5-\sqrt{5}}{3-\sqrt{5}} - \frac{1}{4} = \frac{2}{4(3-\sqrt{5})}$, also $|RM| = \sqrt{\frac{1}{2(3-\sqrt{5})}}$.

$\boxed{4}$

$$R_{Dodekaeder} = \frac{|MK|}{|KL|} = \frac{1}{2}\frac{|MK|}{|KS|} = \frac{1}{2}\frac{|RM|}{|RK|} = \frac{\sqrt{\frac{1}{2(3-\sqrt{5})}}}{1/(2\cdot\sqrt{3})}$$

$$= \sqrt{\frac{3}{2(3-\sqrt{5})}} .$$

Ein nicht verlangtes Fazit zu den Aufgaben $\boxed{1}$ bis $\boxed{4}$:

$$R_{Tetraeder} < R_{Oktaeder} < R_{Würfel} < R_{Ikosaeder} < R_{Dodekaeder}$$

$R_{Tetraeder}$	$R_{Oktaeder}$	$R_{Würfel}$	$R_{Ikosaeder}$	$R_{Dodekaeder}$
$\sqrt{\frac{3}{8}}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{3-\sqrt{5}}}$	$\sqrt{\frac{3}{2(3-\sqrt{5})}}$
0,6124	0,7071	0,8660	0,9510	1,4012