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Nonlinear Optimization (FSS 2023)

Exercise Sheet #9

Due on 07.05.2023 (before 13:00).

1. Tangent Cone [2 points]

Let $X := \{x \in \mathbb{R}^2 : x_1 \geq x_2^2,\ x_2 \geq x_1^2\}$.
Draw a sketch of the feasible region X. Determine $\mathcal{T}\big(X, (0,0)^\top\big)$ and add it to the sketch.

2. Tangent Cone Describes Admissible Directions [4 points]

We introduced the *tangent cone* $\mathcal{T}(X, d)$ to describe "admissible directions". A much more intuitive description for the set of admissible directions woule be given in the set

$$\mathcal{Z}(X, x) = \{d \in \mathbb{R}^n : \exists \bar{\alpha} > 0 : x + \alpha d \in X\ \forall \alpha \in [0, \bar{\alpha}]\}$$

Consider now the constrained optimization problem

$$\begin{aligned} \min f(x_1, x_2) \\ \text{s.t. } x_2 - (x_1^3 + x_1) = 0 \\ x_1 + 1 \geq 0 \end{aligned}$$

with $f(x_1, x_2) = x_2$.

(i) Sketch the feasible region $X \subset \mathbb{R}^2$.

(ii) Determine the tangent cone $\mathcal{T}(X, x)$ w.r.t. the feasible region X for the points $(0,0)^\top$ and $(-1,-2)^\top$, and verify the necessary optimality conditions of Theorem 11.5.

(iii) Determine the set of admissible directions $\mathcal{Z}(X, x)$ for the points $(0,0)^\top$ and $(-1,-2)^\top$. Use the result to explain the necessity of the tangent cone for the formulation of useful optimality conditions.

Hint: You might use $\mathcal{T}(X, x) \subseteq \mathcal{T}_l(g, h, x)$.

3. Constraint Qualification [4 points]

Show the first statement of Theorem 11.9, i.e. show that the following condition is a constraint qualification at $x \in X$: The functions g_i for $i \in \mathcal{A}(x)$ are concave and h is affine linear, i.e.

$$g_i(y) \leq g_i(x) + \nabla g_i(x)^T (y - x) \quad \forall i \in \mathcal{A}(x),\ y \in X \qquad \text{and} \qquad h(x) = Bx - b$$

with $B \in \mathbb{R}^{p \times n},\ b \in \mathbb{R}^p$.

4. Karush-Kuhn-Tucker Conditions [4 points]

Consider the optimization problem

$$\min f(x) \quad \text{s.t.} \quad g(x) = x_1^2 - x_2 \leq 0 \tag{1}$$

with $f(x) = -x_1^2 + 2x_2$.

(a) Show that a constraint qualification is satisfied at $x \in X$.

(b) Use the KKT conditions to determine all candidates for minimizers of (1).

(c) Show that a KKT point from (b) is a minimizer of (1).