University of Mannheim
Scientific Computing, B6 26, C312, D-68131 Mannheim
Dr. Andreas Sommer (ansommer@mail.uni-mannheim.de)

Nonlinear Optimization (FSS 2023)

Exercise Sheet #2

Due on Sunday, 12.03.2023 $\Longrightarrow$ **before 13:00** $\Longleftarrow$

---

1. Exact step size for quadratic functions [4 points]

Let $f(x) = \frac{1}{2}x^\top Qx + b^\top + c$ be a quadratic function with Q symmetric and positive definite, $x \in \mathbb{R}^n$. Then, the exact step size for descent direction $d \in \mathbb{R}^n$ of f in x is

$$\sigma_E = -\frac{\nabla f(x)^\top d}{d^\top Qd}$$

2. Convergence behaviour of the method of steepest descent [4 points]

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a pure quadratic function, i.e. $f(x) = \frac{1}{2}x^\top Qx$, with Q symmetric and positive definite. Consider the method of steepest descent $x^{k+1} = x^k - \sigma_k \nabla f(x^k)$ with a step size control $\sigma_k \geq \sigma_{\min} > 0 \ \ \forall\, k$.

(a) Show for $x^k \neq 0$:

$$\frac{\|x^{k+1}\|}{\|x^k\|} \leq \max\{|1 - \sigma_k \lambda_{min}|, |1 - \sigma_k \lambda_{max}|\}$$

where λ_{min} denotes the smallest and λ_{max} denotes the largest eigenvalue of Q.
[Hint: $\frac{\|x^{k+1}\|}{\|x^k\|} < 1$ is necessary for convergence, and $\sigma > \frac{2}{\lambda_{\max}}$ leads to divergence]

(b) Illustrate the graph of the right hand side of (a), as a function of σ. Can you graphically determine step sizes that lead to convergence?
[Hint: Note that σ_k must not vanish "too quickly"]

3. Admissible directions for Newton-like methods [4 points]

In Newton-like methods, the search directions d^k are determined by linear equation systems

$$M_k d^k = -\nabla f(x^k)\ .$$

Show that if all M_k are symmetric positive definite with

$$0 < \mu_1 \leq \lambda_{min}(M_k) \leq \lambda_{max}(M_k) \leq \mu_2 < \infty$$

(for $\mu_1, \mu_2 > 0$ independent of k), then every subsequence $(d^k)_K$ consists of admissible search directions. (*Hint:* Verify the angle condition for the generated search directions.)