University of Mannheim
Scientific Computing, B6 26, C306, 68131 Mannheim
Dr. Andreas Sommer (ansommer@mail.uni-mannheim.de)

Nonlinear Optimization (FSS 2023)

Exercise Sheet #11

Due on 28.05.2023 (before 13:00).

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1. Derivation of SQP method [4 Points]

Show the derivation #2 of the SQP method, i.e. prove the following statement:

The pair $(d, \mu_{QP}) = (d_x^k, \mu^k + d_\mu^k)$ is a KKT tuple for (13.4) if and only if $d^k = (d_x^k, d_\mu^k)^\top$ is a solution of (13.3).

2. Directional Differentiability [4 Points]

Consider the functions $f_i : \mathbb{R}^2 \to \mathbb{R}$, $i \in \{1, 2\}$ defined by

$$f_1(x_1, x_2) = \begin{cases} \frac{x_1^3}{x_1^2 + x_2^2} & \text{for } (x_1, x_2) \neq (0,0) \\ 0 & \text{for } (x_1, x_2) = (0,0) \end{cases}$$

$$f_2(x_1, x_2) = \begin{cases} \frac{x_1^2}{x_1^2 + x_2^2} & \text{for } (x_1, x_2) \neq (0,0) \\ 1 & \text{for } (x_1, x_2) = (0,0) \end{cases}$$

Visualize (e.g. using Matlab) both functions around the origin.

Show that f_1 is not differentiable but directionally differentiable in all directions $d \in \mathbb{R}^2$ at the origin.

Does this also hold for f_2?

3. The Maratos Effect [4 Points]

Consider the minimization problem

$$\begin{aligned} \min \;\; & f(x) = 2(x_1^2 + x_2^2 - 1) - x_1 \\ \text{s.t.} \;\; & h(x) = x_1^2 + x_2^2 - 1 = 0 \end{aligned}$$

(a) Show that $(\tilde{x}, \tilde{\mu}) = \left((1,0)^\top, -\frac{3}{2}\right)$ is the optimal solution with $\nabla_{xx}^2 \mathcal{L}(\tilde{x}, \tilde{\mu}) = I$.

(b) Show that an iterate $x^k = (\cos\theta, \sin\theta)^\top$ is feasible for every $\theta \in [0, 2\pi]$.

(c) Consider a minimization algorithm that produces a search direction $d^k = \begin{pmatrix} \sin^2\theta \\ -\sin\theta \cdot \cos\theta \end{pmatrix}$.
Show that the step $x^{k+1} := x^k + d^k$ approaches the solution $\tilde{x}$ at a rate consistent to quadratic convergence.
Hint: Determine the ratio $\dfrac{\|x^{k+1} - \tilde{x}\|_2}{\|x^k - \tilde{x}\|_2^2}$.

(d) Calculate $f(x^k)$, $f(x^{k+1})$, $h(x^k)$, and $h(x^{k+1})$. What do you observe? Can the globalized SQP method from the lecture in Algorithm (17) generate these iterates?

4. Active Set Strategy for convex QPs [6 Points]

Use the active set strategy (Algorithm 18) to solve the following problem:

$$\begin{aligned}
\min_{x\in\mathbb{R}^2} \quad & \tfrac{1}{2}(x_1-3)^2 + (x_2-2)^2 && \text{(QP)}\\
\text{s.t.} \quad & -2x_1 + x_2 \le 0 && (1)\\
& x_1 + x_2 \le 4 && (2)\\
& -x_2 \le 0 && (3)
\end{aligned}$$

Starting from the feasible point $x^0 = (0,0)^\top$, for every iteration k, list the iterate x^k, the active set $\mathcal{A}_k$, the associated $(QP)_k$ with respective Lagrangian $\mathcal{L}_k$ and KKT tuple $(\hat{x}^k, \lambda^k, \mu^k)$, the direction d^k, and denote which path/condition/line in the algorithm is taken.

Hint: It holds for the generated iterates: $x^k \in \left\{ \binom{0}{0}, \binom{11/9}{22/9}, \binom{5/3}{7/3}, \binom{7/3}{5/3} \right\}$.

5. Active Set Strategy maintains regularity [4 Points]

Prove that the active set strategy (Algorithm 18) maintains regularity, i.e. if x^k is regular (LICQ hold), then also x^{k+1} is regular.