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Nonlinear Optimization (FSS 2023)

Exercise Sheet #1

Due on Sunday, 19.02.2023, 18:00.

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1. Minima and Maxima [4 points]

Let $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x) = x_1^2 x_2^2 + (x_2^2 - 1)^2$ be given. Find all (global and local) minima, and all (global and local) maxima of f. Which information on minimizers and/or maximizers can be obtained by using first and second order optimality conditions? Use Matlab to plot the function and to check your results.

2. Properties of quadratic functions [2 points]

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a quadratic function, i.e, $f(x) = \frac{1}{2} x^\top Q x + c^\top x + \gamma$ with a symmetric matrix $Q \in \mathbb{R}^{n \times n}$, $c \in \mathbb{R}^n$, $\gamma \in \mathbb{R}$. Show:

(a) f is convex $\Leftrightarrow$ Q is positive semi-definite.

(b) f is strictly convex $\Leftrightarrow$ f is uniformly convex $\Leftrightarrow$ Q is positive definite.

3. Properties of convex functions [4 points]

Let $X \subseteq \mathbb{R}^n$ be open and convex, and let $f : X \to \mathbb{R}$ be $C^1(X)$. Show:

(a) f convex (on X) $\Rightarrow$ $f(x) - f(y) \geq \nabla f(y)^\top (x - y)$ $\forall x, y \in X$.

(b) f strictly convex (on X) $\Rightarrow$ $f(x) - f(y) > \nabla f(y)^\top (x - y)$ $\forall x, y \in X$.

(c) f is uniformly convex (on X) $\Rightarrow$ $\exists \mu > 0$ such that

$$f(x) - f(y) \geq \nabla f(y)^\top (x - y) + \mu \|x - y\|^2 \quad \forall x, y \in X.$$

4. Properties of convex optimization problems (Theorem 3.6) [4 points]

Let $f : X \to \mathbb{R}$ be convex and continuously differentiable on the convex set $X \subseteq \mathbb{R}^n$.
Then, for the optimization problem

$$\min f(x) \quad \text{s.t.} \quad x \in X$$

it holds:

(a) Every local minimizer of f on X is a global minimizer of f on X.

(b) If f is strictly convex, then f has at most one local solution (i.e. it is the unique global solution, if existent).

(c) If X is open and $\tilde{x} \in X$ a stationary point of f, then $\tilde{x}$ is a global minimizer of f on X.