Wahrscheinlichkeitstheorie 1 **Universität Mannheim**
FSS 2021 Prof. L. Döring, Dr. Q. Shi

Sheet 4

For the exercise class 12.04.2021.
Hand in your solutions before 17:00 Saturday 10.04.2021.

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}_0}, \mathbb{P})$ for all the exercises. All the random variables are assumed to be real-valued.

Exercise 1 (Lemma 3.3.1 in Lecture Notes)**.** Let (X_n) be an (F_n)-submartingale and S, T be bounded stopping times such that $S \leq T$ a.s.. Prove that, $\mathbb{E}[X_S] \leq \mathbb{E}[X_T]$.

Exercise 2 (Prop 3.4.4 in Lecture Notes)**.** Let $(X_\alpha, \alpha \in I)$ be a family of random variables with $\sup_{\alpha\in I} \mathbb{E}[|X_\alpha|] < \infty$, i.e. bounded in $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that for every $\epsilon > 0$, there exists $\delta > 0$, such that for all $A \in \mathcal{F}$ with $\mathbb{P}(A) < \delta$, we have

$$\mathbb{E}[|X_\alpha|\mathbf{1}_A] < \epsilon, \qquad \forall \alpha \in I.$$

Prove that $(X_\alpha, \alpha \in I)$ is uniformly integrable.

Exercise 3 (Theorem 3.4.6 in Lecture Notes: Scheffé's Lemma)**.** Suppose that $X_n, n \geq 1$ is a sequences of random variables in $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $\lim_{n\to\infty} X_n = X_\infty$ in probability. Show that the following statements are equivalent:

(ii) $X_n \to X_\infty$ in $\mathcal{L}^1$, as $n \to \infty$;

(iii) $\mathbb{E}[|X_n|] \to \mathbb{E}[|X_\infty|] < \infty$, as $n \to \infty$.

Hint: you may use the fact $|X_n - X_\infty| = |X_n| + |X_\infty| - 2|X_n \wedge X_\infty|$

Exercise 4 (Galton–Watson Process with finite second moment; c.f. Example 3.2.5 in lecture notes)**.** Let $(X_n, n \in \mathbb{N}_0)$ be a branching process, starting from $X_0 = m \in \mathbb{N}_0$, with offspring distribution $(p_k, k \in \mathbb{N}_0)$. Recall that, with $(\xi_i^{(n)}, i \geq 1, n \geq 0)$ i.i.d. distributed according to the offspring distribution (i.e. $\mathbb{P}(\xi = k) = p_k$), we can define (X_n) by iteration:

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)}, \qquad n \geq 1.$$

Suppose that $\mu = \mathbb{E}[\xi_1^{(0)}] = \sum_{k=1}^\infty kp_k < \infty$ and $\eta := \mathbb{E}[(\xi_1^{(0)})^2] < \infty$ (the offspring distribution has finite second moment).

(i) Let $f(s) := \mathbb{E}[s^{\xi_1^{(0)}}] = \sum_{k=0}^\infty p_k s^k$, $|s| \leq 1$ be the generating function of $\xi_1^{(0)}$. Show that, the generating function of X_n (i.e. $\mathbb{E}[s^{X_n}] = \sum_{k=0}^\infty \mathbb{P}(X_n = k)s^k$) is given by the iteration relation of function composition:

$$f_n(s) = f(f_{n-1}(s))$$

(ii) Using the fact that

$$f''_{n+1}(s) = f''(f_n(s))[f'_n(s)]^2 + f'(f_n(s))f''_n(s),$$

prove that

$$f''_n(1) = f''(1)\left(\mu^{2n-2} + \mu^{2n-1} + \cdots + \mu^{n-1}\right).$$

(iii) Prove that

$$\mathbb{E}[X_n^2] = \frac{(\eta - \mu)\mu^{n-1}(\mu^n - 1)}{\mu - 1} + \mu^n, \text{ if } \mu \neq 1$$

and

$$\mathbb{E}[X_n^2] = n(\eta - 1) + 1, \text{ if } \mu = 1.$$

(iv) Recall that $M_n := \mu^{-n}X_n$ is a martingale and $M_n \to M_\infty$ a.s.. If $\mu > 1$, then show that $M_n \to M_\infty$ in $\mathcal{L}^2$ and $\mathbb{E}[M_\infty] = 1$.

Exercise 5. ABRACADABRA

At each of times $1, 2, \ldots$, a monkey types a capital letter at random, the sequence of letters typed forming an i.i.d. sequence of random variables chosen uniformly amongst the 26 possible capital letters. Let T be the first time by which the monkey has produced the consecutive sequence ABRACADABRA. We prove that

$$\mathbb{E}[T] = 26^{11} + 26^4 + 26.$$

For this, imagine that just before **each** time $n \in \mathbb{N}$, a new gambler indexed by n arrives on the scene. He bets 1 euro that

the n-th letter will be A.

If he loses, he loses all his money and leaves. If he wins, he receives 26 euros, all of which he bets on the event that

the $n+1$-th letter will be B.

If he loses, he loses all his money and leaves. If he wins, he bets his whole current fortune of 26^2 that

the $n+2$-th letter will be R.

and so on through the ABRACADABRA sequence. All gamblers come and play and leave independently of each other. Let $(X_i^k, i \geq 0)$ denote the fortune of the k-th gambler just after time i: in particular, $X_i^k = 1$ for all $i \leq k-1$.

(i) Show that indeed, $\mathbb{E}[T] < \infty$. **Hint:** See exercise 2 in Sheet 3.

(ii) For every $n \geq 1$, let M_n be the total net profit of all gamblers who have participated in the game just after time n, i.e.

$$M_n = \sum_{k=1}^{n}(X_n^k - 1),$$

and $M_0 := 0$.

Show that $(M_n)_{n\geq 0}$ is a martingale.

(iii) Show that $T = \inf\{n \geq 0 : M_n + n \geq 26^{11} + 26^4 + 26\}$. Conclude that $\mathbb{E}[T] = 26^{11} + 26^4 + 26$.