Wahrscheinlichkeitstheorie 1
FSS 2022

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Sheet 8

For the exercise class on the 25.04.2022.
Hand in your solutions by 10:15 in the exercise on Monday 25.04.2022.

Exercise 1 (Complex Integration)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $f, g\colon \Omega \to \mathbb{C}$ an integrable function. Show that

(i) $\int_\Omega af + g \,\mathrm{d}\mu = a\int_\Omega f \,\mathrm{d}\mu + \int_\Omega g \,\mathrm{d}\mu$, for all $a \in \mathbb{C}$

(ii) $\mathrm{Re}(\int_\Omega f \,\mathrm{d}\mu) = \int_\Omega \mathrm{Re}(f) \,\mathrm{d}\mu$, $\mathrm{Im}(\int_\Omega f \,\mathrm{d}\mu) = \int_\Omega \mathrm{Im}(f) \,\mathrm{d}\mu$

(iii) $\overline{\int_\Omega f \,\mathrm{d}\mu} = \int_\Omega \overline{f} \,\mathrm{d}\mu$

Exercise 2 (Characteristic Functions)**.**

(i) Let X be a random variable on $\mathbb{R}^d$ with characteristic function φ_X. For $a \in \mathbb{R}, b \in \mathbb{R}^d$, show that the characteristic function of $aX + b$ is $\varphi_{aX+b}(t) = \varphi_X(at)e^{i\langle b,t\rangle}$.

(ii) Let X be a random variable on $\mathbb{R}^d$ and Y be a random variable on $\mathbb{R}^k$. Prove that, X is independent of Y, if

$$\mathbb{E}\left[\exp\Big(i\big(\langle t, X\rangle + \langle s, Y\rangle\big)\Big)\right] = \varphi_X(t)\varphi_Y(s), \qquad \forall t \in \mathbb{R}^d, s \in \mathbb{R}^k.$$

Hint. *The characteristic function determines the distribution, uniquely! Pick a different random vector* $(\tilde{X}, \tilde{Y})$ *with* $\tilde{X} \overset{(d)}{=} X$ *and* $\tilde{X} \overset{(d)}{=} X$ *and assume independence.*

Exercise 3 (One-Point Compactification)**.** We consider the space $([0, \infty], d)$ with

$$d(x, y) := |e^{-x} - e^{-y}|.$$

Prove that

(i) d is a metric

(ii) and the space is compact.

Hint. *Sequences.*

Exercise 4 (Gaussian Vectors)**.** For any $k \geq 1$, a random variable $Z := (Z_1, \ldots, Z_k)$ on $\mathbb{R}^k$ is called a *centred Gaussian vector*, if every linear combination of $Z_1, \ldots, Z_k$ is a centred Gaussian random variable on $\mathbb{R}$.

(i) Prove that, Z is a centred Gaussian vector, if and only if

$$\mathbb{E}\left[\exp(i\langle t, Z\rangle)\right] = \exp\left(-\frac{1}{2}t^T M t\right), \qquad \forall t \in \mathbb{R}^k,$$

where M is a $k \times k$ matrix, with $M_{ij} = \mathrm{Cov}(Z_i, Z_j)$.

Hint. *recall that the characteristic function of* $\mathcal{N}(0, \sigma^2)$ *is* $\varphi(t) = \exp(-\frac{1}{2}\sigma^2 t^2)$

(ii) Let (X, Y) be a centred Gaussian vector. Prove that, X is independent of Y, if and only if $\text{Cov}(X, Y) = 0$.

Exercise 5 (Complex Analysis)**.** (Optional Difficult Easter Challenge)
Read about path integrals in Complex Analysis (Funktionentheorie). In particular Cauchy's Integral Theorem.

(i) To calculate an integral over the path $\gamma_1 : [0, 1] \to \mathbb{C}, s \mapsto -(n + it) + 2sn$ you could therefore find a path γ_2 which connects $n - it$ to $-n - it$. Because connecting both paths together results in a closed loop, we get by Cauchy's integral theorem

$$\int_{-n}^{n} f(y - it)dy \overset{x=y-it}{=} \int_{\gamma_1} f(x)dx = -\int_{\gamma_2} f(x)dx.$$

Use this insight to calculate the characteristic function of the standard normal distribution.

Hint. *Draw boxes in* $\mathbb{C}$ *considering*

$$\begin{aligned} \varphi_X(t) &= \frac{1}{\sqrt{2\pi}} \int e^{itx} e^{-\frac{x^2}{2}} \, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{(x-it)^2}{2}} e^{-\frac{t^2}{2}} \, \mathrm{d}x \\ &= e^{-\frac{t^2}{2}} \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \int_{-n}^{n} e^{-\frac{(x-it)^2}{2}} \, \mathrm{d}x. \end{aligned}$$

(ii) Use a similar approach to prove that $\varphi_X(t) = \frac{1}{1-it}$ for $X \sim \text{Exp}(1)$.

Hint. *Here simple boxes will not be enough. A particularly interesting path is the linear path connecting* $(1 - it)a$ *to* $(1 - it)b$*. Mind the derivative when substituting!*