Wahrscheinlichkeitstheorie 1 | **Universität Mannheim**
FSS 2022 | Prof. Leif Döring, Felix Benning

Sheet 7

For the exercise class on the 04.04.2022.
Hand in your solutions by 10:15 in the exercise on Monday 04.04.2022.

Exercise 1 (Portmonteau)**.** True of false? **Justify your answer** by proving or disproving the statement. Let $(\mu_n, n \geq 1), \mu$ be finite measures (not necessarily probability measure) on $\mathbb{R}$.

(i) If $\mu_n \to \mu$ weakly, then $\lim_{n\to\infty} \mu_n(\mathbb{R}) = \mu(\mathbb{R})$.

(ii) If $\mu_n \to \mu$ weakly, then $\lim_{n\to\infty} \mu_n((-\infty, x]) = \mu((-\infty, x])$ for all $x \in \mathbb{R}$.

(iii) If $\lim_{n\to\infty} \mu_n((-\infty, x]) = \mu((-\infty, x])$ for all $x \in \mathbb{R}$, then $\mu_n \to \mu$ weakly.

(iv) Suppose that $\mu_n, \mu \in \mathcal{M}_1(\mathbb{R})$. If $\lim_{n\to\infty} \mu_n((-\infty, x]) = \mu((-\infty, x])$ for all $x \in \mathbb{R}$, then $\mu_n \to \mu$ weakly.

(v) Let f be a measurable bounded function on $\mathbb{R}$ which is μ-a.e. continuous. If $\mu_n \to \mu$ weakly, then $\int_\mathbb{R} f \mathrm{d}\mu_n \to \int_\mathbb{R} f \mathrm{d}\mu$.

Exercise 2 (Tightness)**.** Let $(\mu_n, n \geq 1)$ be finite measures on $\mathbb{R}$. Show that the family is tight, if and only if there exists a non-negative measurable function f with $\lim_{|x|\to\infty} f(x) = \infty$, such that

$$\sup_{n\geq 1} \int_\mathbb{R} f \mathrm{d}\mu_n < \infty.$$

Exercise 3 (Totally Bounded)**.** Let (E, d) be a metric space and $A \subset E$. Compare Lemma 4.3.4. with this exercise.

(i) Suppose that, for all $\epsilon > 0$, there exists finitely many points $x_1, \dots x_n \in E$ (not necessarily in A) such that $A \subseteq \bigcup_{k=1}^n (B_{\epsilon,k})$. Show that A is totally bounded.

(ii) Let $E = \mathbb{Q}$, endowed with the usual Euclidean metric. Show that, $A = \mathbb{Q} \cap (0, 1)$ is totally bounded, but not relatively compact. Why is this not a contradiction to Sheet 6, Exercise 2(iv)?

Exercise 4 (Weak Convergence of Dirac Measures)**.** Let $x_n \in E$, where (E, d) is a separable and complete metric space.

(i) Prove that δ_{x_n} is tight, if and only if $(x_n)_{n\in\mathbb{N}}$ is totally bounded in E.

(ii) For $E = \mathbb{R}$ suppose that δ_{x_n} converges weakly to some μ. Show that, there exists $x \in \mathbb{R}$, such that $x_n \to x$ and therefore $\mu = \delta_x$.

Hint. *Assume you could use the identity as a test function in weak convergence. While this does not work (the identity is not bounded) it provides some intuition how the actual proof would work. Then try an invertible sigmoid function instead (e.g. $f(x) = \frac{1}{1+e^{-x}}$)*

Exercise 5 (Arzelà-Ascoli)**.** We are going to prove the Arzelà-Ascoli theorem in this exercise. While it is usually formulated with (relative) compactness, we have already seen this is equivalent to total boundedness in a complete space.

Definition (Equicontinuity)**.** *Let $(E,d),(F,d)$ be metric spaces and $\mathcal{F}$ a set of functions from E to F. Then $\mathcal{F}$ is* equicontinuous *iff*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall f \in \mathcal{F} : d(x,y) < \delta \implies d(f(x), f(y)) < \varepsilon.$$

In other words: δ is independent of $f \in \mathcal{F}$ (and also of x, y, i.e. we also have uniform continuity).

(i) Prove that any finite subset of $C(K)$ for a compact space K is equicontinuous.

(ii) Prove that total boundedness of a subset of $(C(K), \|\cdot\|_\infty)$ implies equicontinuity.

(iii) Let $\mathcal{F} \subseteq C(K)$ be a *pointwise bounded* set, i.e.

$$\forall x \in K, \exists M(x) > 0 : |f(x)| < M(x) \quad \forall f \in \mathcal{F}.$$

Prove that for any finite set $\{x_1, \ldots, x_d\} \subseteq K$, the set

$$\Phi = \{(f(x_1), \ldots, f(x_d)) : f \in \mathcal{F}\}$$

is totally bounded.

(iv) Prove that pointwise boundedness and equicontinuity of $\mathcal{F} \subseteq C(K)$ imply total boundedness (with regard to the sup-norm $\|\cdot\|_\infty$).

Hint. *Use equicontinuity to extend (iii) to all points. Do not forget that K is totally bounded!*

Putting (ii) and (iv) together we therefore have

Theorem (Arzelà-Ascoli)**.** *Let $K \subseteq E$ be totally bounded in a metric space (E,d). A set of continuous functions mapping to $\mathbb{R}$, $\mathcal{F} \subseteq C(K)$, is totally bounded with regard to the sup-norm iff it is pointwise bounded and equicontinuous.*

(v) Deduce that $\mathcal{P} := \{x \mapsto x^n : n \in \mathbb{N}\} \subseteq C([0,1])$ is bounded but not totally bounded.