Wahrscheinlichkeitstheorie 1 | **Universität Mannheim**
FSS 2022 | Prof. Leif Döring, Felix Benning

Sheet 12

For the exercise class on the 23.05.2022.
Hand in your solutions by 10:15 in the exercise on Monday 23.05.2022.

Exercise 1 (Brownian Motion Unbounded)**.** Prove that the Brownian motion is unbounded without using Exercise 3.

Hint. *Borel Cantelli.*

Exercise 2 (Brownian Bridge)**.** A stochastic process $(X_t)_{0\leq t\leq 1}$ is called a *Brownian bridge* if it is a continuous centred Gaussian process with covariance function given by

$$\mathrm{Cov}(X_s, X_t) = s(1-t), \quad s \leq t.$$

Let $(B_t, t \geq 0)$ be a standard Brownian motion. Show that the process

$$X_t = B_t - tB_1, \quad t \in [0,1]$$

is a Brownian bridge. Moreover, prove that B_1 is independent of $(X_t, t \in [0,1])$.

Exercise 3 (Brownian Growth Rate)**.** Let $(B_t \geq 0)$ be a Brownian motion.

(i) Prove that, for any $k > 0$, we have

$$\mathbb{P}(\inf\{t > 0\colon B_t \geq k\sqrt{t}\} = 0) = 1.$$

Hint. *Look for a similar statement in the lecture notes.*

(ii) Prove that,

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{t}} = \infty \quad \text{and} \quad \liminf_{t\to\infty} \frac{B_t}{\sqrt{t}} = -\infty \quad \text{a.s..}$$

Hint. *Use time inversion.*

Exercise 4 (Compound Poisson is Lévy)**.** Show that the compound Poisson process $X = (X_t)_{t\geq 0}$ is a Lévy Process. You can assume that the Poisson process $N = (N_t)_{t\geq 0}$ is a Lévy Process.

Hint. *Show* $\mathbb{E}[f(X_{\overline{t}} - X_{\underline{t}})g(X_{\overline{s}} - X_{\underline{s}})]$ *factorizes.*