Wahrscheinlichkeitstheorie 1 FSS 2022

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Sheet 10

For the exercise class on the 09.05.2022.
Hand in your solutions by 10:15 in the exercise on Monday 09.05.2022.

Exercise 1. Show that the set of all finitely generated cylinder sets form a semiring $\mathcal{Q}$.

Hint. *First draw some pictures and then try to formalize your findings using projections.*

Exercise 2 (Multivariate Central Limit Theorem)**.** Let $X_1, X_2, \dots$ be iid random vectors with $\mu = \mathbb{E}[X_1]$, $\Sigma = \mathrm{Cov}(X_1)$. Then prove that

$$M_n := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (X_k - \mu) \to \mathcal{N}(0, \Sigma) \quad (n \to \infty)$$

Hint. *Consider* $\langle M_n, t \rangle$.

Exercise 3 (Markov Chain Existence)**.** Let $p(x, y) = \kappa(x, \{y\})$ be a discrete probability kernel on $\mathbb{R}^d$, p_0 a discrete probability distribution. Prove that there exists a stochastic process X_n, such that

(i) $\mathbb{P}(X_n = y \,|\, X_{n-1} = x) = p(x, y)$ and $\mathbb{P}(X_0 = x) = p_0(x)$

(ii) X_n is a Markov Chain, i.e.

$$\mathbb{P}(X_{n+1} = x_{n+1} \,|\, X_n = x_n, \dots, X_0 = x_0) = \mathbb{P}(X_{n+1} = x_{n+1} \,|\, X_n = x_n).$$

Hint. *Consider*

$$\mathbb{P}_{X_0,\dots,X_{n+1}}(A_0, \dots, A_{n+1}) := \sum_{x_0 \in A_0} \cdots \sum_{x_n \in A_n} \kappa(x_n, A_{n+1}) p(x_{n-1}, x_n) \dots p(x_0, x_1) p_0(x_0).$$

and prove that $\mathbb{P}_{X_0,\dots,X_{n+1}}(A_0, \dots, A_n, \mathbb{R}^d) = \mathbb{P}_{X_0,\dots,X_n}(A_0, \dots, A_n)$. *Then construct a consistent family of distributions.*

Exercise 4 (Positive semi-definite Functions)**.** Prove the following statements

(i) Let $\varphi : [0, \infty) \to \mathbb{R}$ be a continuous convex function with $\lim_{x\to\infty} \varphi(x) = 0$. Then $k(s, t) = \varphi(|t - s|)$ is positive semi-definite for all $s, t \in \mathbb{R}$.

Hint. *Pòlya's Theorem.*

(ii) $k(s, t) = e^{-|t-s|}$ is positive semi-definite for all $s, t \in \mathbb{R}$.

Another trick to check positive semi-definiteness is based around scalar products in an arbitrary hilbertspace.

Remark. *This is closely related to "kernels" from support vector machines, where you want to separate data with a "hyper-surface". For the surface to be not necessarily a hyperplane, you first map the data to some hilbertspace and look for a separating hyperplane in that space instead – recall that a hyperplane can be parametrized as $\{x \in H : \langle x, a \rangle = c\}$, where a is the normal vector.*

(iii) Let H be a complex hilbertspace, $\phi : M \to H$ any function. Then $k(x, y) = \langle \phi(x), \phi(y) \rangle$ is positive semi-definite for all $x, y \in M$.

(iv) $k(s, t) = e^{i(s-t)}$ is positive semi-definite for all $s, t \in \mathbb{R}$.

(v) $k(s, t) = \cos(s - t)$ is positive semi-definite for all $s, t \in \mathbb{R}$.

And sometimes you just have to apply the definition manually.

(vi) $k(s, t) = \min\{s, t\}$ is positive semi-definite for $s, t \in [0, \infty)$.

Hint. *Sort the t_i and start induction with $n = 2$.*