Sheet 6

For the exercise class on the 28.03.2022.
Hand in your solutions at 10:15 in the exercise on Monday 28.03.2022.

Exercise 1 (Continuous Indicator)**.** Consider a metric space (E, d). For $x \in E$ and $B \subseteq E$, define

$$d(x, B) := \inf\{d(x, y) \colon y \in B\}.$$

(i) If $d(x, B) = 0$, then $x \in \bar{B}$, the closure of B.

(ii) Suppose that $B \neq \emptyset$. Prove that

$$|d(x, B) - d(y, B)| \leq d(x, y).$$

This is to say, the function $x \mapsto d(x, B)$ is 1-Lipschitz.

(iii) Let $A \subseteq E$ be **closed**. Show that, for any $\epsilon > 0$, the function

$$f_A^\epsilon(x) = (1 - d(x, A)/\epsilon)^+$$

satisfies the following properties:

- $\forall x \in A$, $f_A^\epsilon(x) =$1
- if $d(x, A) \geq \epsilon$, then $f_A^\epsilon(x) =$0
- $f_A^\epsilon \to \mathbb{1}_A$ pointwisely
- $f_A^\epsilon \leq$1
- $f_A^\epsilon \in \mathrm{Lip}_{1/\epsilon}(E)$ and $f_A^\epsilon \in C_b(E)$

Why do we have to assume A to be closed?

Exercise 2 (Totally Bounded Spaces)**.**

(i) Show that totally bounded sets are always bounded.

(ii) Prove that total boundedness and boundedness are equivalent on $\mathbb{R}^d$ for any metric induced by a norm.

(iii) Prove that no infinite set is totally bounded in the discrete metric

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$

(iv) Prove that the closure of a totally bounded set is totally bounded.

Remark. *In a complete metric space totally bounded is therefore by Lemma 7.1.8 equivalent to relatively compact.*

Exercise 3 (Non-separable Space)**.** Consider $\mathbb{R}$ endowed with the Lebesgue measure $\texttt{Leb}$. Let L_∞ be the space of measurable functions that are bounded almost everywhere, with the essential supremum of its absolute value as a norm: for any measurable function f,

$$\|f\|_\infty = \inf\{C \geq 0 \colon \texttt{Leb}(f > C) = 0\}.$$

Show that the metric space L^∞ (with metric induced by the norm $\|\cdot\|_\infty$) is not separable.

Hint. consider all the indicator function $\mathbf{1}_{(-r,r)}$, $r >$0

Exercise 4 (Weak Limits)**.** Find the weak limit, if exists, of the following sequence.

(i) Assume $x_n \to x$ in some metric space (E, d). Prove that $\delta_{x_n} \to \delta_x$.

(ii) $\mathbb{P}_n = \mathcal{N}(0, 1/n)$, the normal distribution.

Hint. What does weak convergence have to do with random variables?

Exercise 5 (Borel σ-Algebra)**.** Let (E, d) be a metric space with Borel σ-Algebra

$$\mathcal{B}(E) := \sigma(\{O \subseteq E : O \text{ open}\})$$

(i) Assume there exists a countable base π of the topology τ of E. Prove that $\sigma(\pi) = \mathcal{B}(E)$.

(ii) Assume that E is separable (has a countable dense subset A). Prove that

$$\pi := \{B_q(x) : x \in A, q \in \mathbb{Q}\}$$

is a (countable) base of τ.

Remark. *In case of $\mathbb{R}$, this base π is the set of all intervals with rational endpoints.*

Hint. $\bigcup\{B \in \pi : B \subseteq O\} = O$

(iii) Prove that $\sigma(\{B_\epsilon(x), x \in E, \epsilon > 0\}) = \mathcal{B}(E)$ if E is separable.