Wahrscheinlichkeitstheorie 1 | **Universität Mannheim**
FSS 2022 | Prof. Leif Döring, Felix Benning

Sheet 5

For the exercise class on the 21.03.2022.
Hand in your solutions at 10:15 in the exercise on Monday 21.03.2022.

Exercise 1 (Uniform Integrability)**.** Show that the following families are uniformly integrable:

(i) (Prop. 6.3.14 (ii)⇒(i)) $(X_\alpha, \alpha \in I)$ a family of random variables with $\sup_{\alpha \in I} \mathbb{E}[|X_\alpha|] < \infty$, i.e. bounded in $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that for every $\epsilon > 0$, there exists $\delta > 0$, such that for all $A \in \mathcal{F}$ with $\mathbb{P}(A) < \delta$, we have

$$\mathbb{E}[|X_\alpha| \mathbf{1}_A] < \epsilon, \qquad \forall \alpha \in I.$$

(ii) a sequence of identically distributed random variables $\{X_i, i \in \mathbb{N}\}$ with $\mathbb{E}[|X_1|] < \infty$.

(iii) Let $(X_i, i \in \mathbb{N})$ be a sequence of i.i.d. random variables with $\mathbb{E}[|X_1|] < \infty$. Let $S_n = \sum_{i=1}^n X_i$ for every $n \in \mathbb{N}$. Then the family $\{\frac{1}{n} S_n, n \in \mathbb{N}\}$ is uniformly integrable.

Exercise 2 (Not Uniformly Integrable)**.** Let $(U_n, n \geq 1)$ be i.i.d. Bernoulli random variables with $\mathbb{P}(U_n = 1) = p \in (0, 1)$ and $\mathbb{P}(U_n = 0) = 1 - p$. Set $\tau := \inf\{n \geq 1 \colon U_n = 1\}$. Define $X_n := (1 - p)^{-n} \mathbb{1}_{\tau > n}$.

(i) Show that (X_n) is a martingale (choose a suitable filtration).

(ii) Prove that $X_n \to 0$ almost surely

(iii) Prove that $(X_n, n \geq 1)$ is not uniformly integrable.

Exercise 3 (Tail σ-Algebra)**.**

(i) $\mathcal{A}$ is trivial if an only if all $(\mathcal{A}, \mathcal{B}(\mathbb{R}^d))$-measurable functions are constant

Let $X_1, X_2, \dots$ be a sequence of random variables with $\tau_n = \sigma(X_n, \dots)$ and tail σ-algebra $\tau = \bigcap_n \tau_n$.

(ii) Prove that $\liminf_n X_n$ and $\limsup_n X_n$ are τ measurable.

(iii) Prove that

$$C := \{\omega \in \Omega : \lim_{n \to \infty} X_n \text{ exists}\} \in \tau$$

Exercise 4 (Gambler's ruin)**.** Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{P}(X_1 = 1) = 1 - \mathbb{P}(X_1 = -1) = p$ for some $p \in (0, 1)$, $p \neq 1/2$. Let $a, b \in \mathbb{N}$ with $0 < a < b$. Define $S_0 := a$ and for every $n \geq 1$, $S_n := S_{n-1} + X_n$. Finally, define the following stopping time:

$$T := \inf\{n \geq 0 : S_n = 0 \text{ or } S_n = b\}.$$

We consider the filtration generated by $(X_n)_{n \geq 1}$.

(i) Show that $\mathbb{E}[T] < \infty$.

(ii) Consider for every $n \geq 0$,

$$M_n := \left(\frac{1-p}{p}\right)^{S_n} \quad \text{and} \quad N_n := S_n - n(2p-1).$$

Prove that $(M_n)_{n\geq 0}$ and $(N_n)_{n\geq 0}$ are martingales.

(iii) Deduce the values of $\mathbb{P}(S_T = 0)$ and $\mathbb{P}(S_T = b)$.

(iv) Compute the value of $\mathbb{E}[T]$.

(v) Let $\tau_0 := \inf\{n \geq 0 : S_n = 0\}$ and $\tau_b := \inf\{n \geq 0 : S_n = b\}$. Compute the value of $\mathbb{P}(\tau_0 < \infty)$.

Hint. Consider $\mathbb{P}(\tau_0 < \tau_b)$ and let $b \to \infty$.