Sheet 4

For the exercise class on the 14.03.2022.
Hand in your solutions at 10:15 in the exercise on Monday 14.03.2022.

Exercise 1 (Scheffé's Lemma)**.** Suppose that $(X_n, n \geq 1)$ is a sequences of random variables in $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $\lim_{n\to\infty} X_n = X_\infty$ almost surely. Show that the following statements are equivalent:

(i) $X_n \to X_\infty$ in $\mathcal{L}^1$, as $n \to \infty$;

(ii) $\mathbb{E}[|X_n|] \to \mathbb{E}[|X_\infty|] < \infty$, as $n \to \infty$.

Hint. Apply Fatou's Lemma to $|X_n| + |X_\infty| - |X_n - X_\infty| \geq 0$.

Exercise 2 (Lemma 6.3.5)**.** Let (X_n) be an $(\mathcal{F}_n)$-submartingale and S, T be bounded stopping times such that $S \leq T$ a.s. Prove that, $\mathbb{E}[X_S] \leq \mathbb{E}[X_T]$.

Hint. Consider the stochastic integral of $H = (H_n)$ against X with $H_n := \mathbb{1}_{S \leq n-1 < T}$.

Exercise 3 (Uniform Integrability)**.**

(i) (Example 6.3.12 (i)) Prove that every finite set of integrable random variables is uniformly integrable.

(ii) If $X_\alpha \sim \delta_{x_\alpha}$, then the set $(X_\alpha)_{\alpha \in I}$ is uniformly bounded if and only if $S = \{x_\alpha : \alpha \in I\} \subseteq \mathbb{R}$ is uniformly bounded.

Exercise 4 (Uniqueness of Limits in Probability)**.**

(i) (Triangle Inequality of Probability) Prove that for X, Y, Z random variables on some metric space with metric d we have

$$\{d(X,Y) \geq \epsilon\} \subseteq \{d(X,Z) \geq \tfrac{\epsilon}{2}\} \cup \{d(Z,Y) \geq \tfrac{\epsilon}{2}\}$$

Let us now assume that $X_n \xrightarrow{P} X$ and $X_n \xrightarrow{P} Y$

(ii) Prove that $P(d(X,Y) \geq \epsilon) = 0$ for all $\epsilon > 0$.

Hint. Recall the proof of uniqueness for deterministic sequences. Apply a similar trick.

(iii) Conclude that $X = Y$ a.s.

Exercise 5 (Bound on Stopping Times)**.** Let T be a $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$-stopping time. Suppose that there exists $N \in \mathbb{N}$ and $\varepsilon \in (0,1)$, such that for every $n \in \mathbb{N}$,

$$\mathbb{P}(T \leq n + N | \mathcal{F}_n) > \varepsilon, \quad \text{almost surely.}$$

(i) Prove that for each $k \in \mathbb{N}$, we have $\mathbb{P}(T > kN) \leq (1-\varepsilon)^k$.

Hint. $\mathbb{P}(T > kN) = \mathbb{P}(T > kN, T > (k-1)N)$.

(ii) Deduce that $\mathbb{E}[T] < \infty$.

Exercise 6 (Monkey Typewriter Theorem)**.** At each of times $1, 2, \ldots$, a monkey types a capital letter at random, the sequence of letters typed forming an i.i.d. sequence of random variables chosen uniformly amongst the 26 possible capital letters. Let T be the first time by which the monkey has produced the consecutive sequence ABRACADABRA of 11 letters. We prove that

$$\mathbb{E}[T] = 26^{11} + 26^4 + 26.$$

For this, imagine that just before **each** time $n \in \mathbb{N}$, a new gambler indexed by n arrives on the scene. He bets 1 euro that

the n-th letter will be A.

If he loses, he loses all his money and leaves. If he wins, he receives 26 euros, all of which he bets on the event that

the $n+1$-th letter will be B.

If he loses, he loses all his money and leaves. If he wins, he bets his whole current fortune of 26^2 that

the $n+2$-th letter will be R.

and so on through the ABRACADABRA sequence. All gamblers come and play and leave independently of each other. Let $(X_i^k, i \geq 0)$ denote the fortune of the k-th gambler just after time i: in particular, $X_i^k = 1$ for all $i \leq k-1$.

(i) Show that indeed, $\mathbb{E}[T] < \infty$.

Hint. See Exercise 5.

For simplicity of notation, we can add infinitely many Z's after ABRACADABRA. Gamblers then bet on the word ABRACADABRAZZZZZ... We denote by L_k the k-th letter of this word for every $k \geq 1$. We also denote by U_n the n-th letter typed for every $n \geq 1$ and $(\mathcal{F}_n)_{n\geq 0}$ the corresponding filtration.

(ii) For every $n \geq 1$, let M_n be the total net profit of all gamblers who have participated in the game just after time n, i.e.

$$M_n = \sum_{k=1}^{n} (X_n^k - 1),$$

and $M_0 := 0$. Show that $(M_n)_{n\geq 0}$ is a martingale.

Hint. Show that $M_n^k := X_n^k - 1$ is a martingale for every k, by defining $M_n^k := 0$ for all $n < k$.

(iii) Show that $T = \inf\{n \geq 0 : M_n + n \geq 26^{11} + 26^4 + 26\}$. Conclude that $\mathbb{E}[T] = 26^{11} + 26^4 + 26$.