Wahrscheinlichkeitstheorie 1 | **Universität Mannheim**
FSS 2022 | Prof. Leif Döring, Felix Benning

Sheet 3

For the exercise class on the 7.03.2022.
Hand in your solutions at 13:30 in the exercise on Monday 7.03.2022.

Exercise 1 (Martingales Warm-up)**.** Do enough of these exercises to feel comfortable with them.

(i) Assume that X_n is an integrable $(\mathcal{F}_n)$ adapted stochastic process which is decreasing, i.e.

$$X_{n+1} \leq X_n \quad \text{a.s..}$$

Prove that X_n is a supermartingale.

Remark. *Similarly an adapted increasing process is a submartingale.*

(ii) Let $(X_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$ be $\mathcal{F}_n$ (sub-)martingales, prove that

(a) $(X_n + Y_n)_{n\in\mathbb{N}}$ is a (sub-)martingale.

(b) $(X_n \vee Y_n)_{n\in\mathbb{N}}$ is a submartingale,

(iii) Let $(X_n)_{n\in\mathbb{N}_0}$ be a $(\mathcal{F}_n)$-submartingale. Let T be $(\mathcal{F}_n)$-stopping time.

(a) Suppose that $(X_n)_{n\in\mathbb{N}_0}$ is bounded and T is a.s. finite (the meaning is different from "T is a.s. bounded"!). Prove that $\mathbb{E}[X_0] \leq \mathbb{E}[X_T] < \infty$.

(b) Suppose that there exists a constant $K > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$|X_n(\omega) - X_{n-1}(\omega)| \leq K, \forall n \in \mathbb{N}.$$

We also suppose that $\mathbb{E}[T] < \infty$. Show that $\mathbb{E}[X_0] \leq \mathbb{E}[X_T] < \infty$.

Hint. Compare this with statements about martingales in the lectures and use the bounded stopping time $T \wedge M$ for some constant $M \in \mathbb{N}$ and the dominated convergence theorem.

Exercise 2 (Conditional Expectation)**.** Recall that a gamma distribution with parameter $c > 0$ and $\theta > 0$ has density:

$$\frac{\theta^c}{\Gamma(c)} x^{c-1} e^{-\theta x} \mathbb{1}_{x>0}.$$

(i) Let X, Y be two independent exponential random variables with parameter $\theta > 0$ and $Z = X + Y$. Determine the conditional distribution of X given $Z = z$ (i.e. the Markov kernel $(A, z) \mapsto \mathbb{P}(X \in A \mid Z = z)$).

(ii) Conversely, let Z be a random variable with gamma distribution with parameter $(2, \theta)$, and suppose X is a random variable whose conditional distribution given $Z = z$ is uniform on $[0, z]$, for $z > 0$. Prove that X and $Z - X$ are independent with exponential distribution $\texttt{Exponential}(\theta)$.

Exercise 3 (Branching Process Tools)**.** Assume $X = (X_n)_n$ are identically distributed, independent and integrable random variables. And assume N is an integrable $\mathbb{N}_0$-valued random variable independent of X and $S_n := \sum_{k=0}^n X_i$.

(i) (Wald's equation) Prove that

$$\mathbb{E}[S_N] = \mathbb{E}[N]\mathbb{E}[X_1]$$

Hint. Use indicators and Fubini.

(ii) (Blackwell-Girshick) Further assume that $X_n, N \in \mathcal{L}^2$. Prove that

(a) $\mathbb{E}[S_N^2] = \mathbb{E}[N]\mathrm{Var}[X_1] + \mathbb{E}[N^2]\mathbb{E}[X_1]^2$

(b) $\mathrm{Var}[S_N] = \mathbb{E}[N]\mathrm{Var}[X_1] + \mathrm{Var}[N]\mathbb{E}[X_1]^2$

Hint. Same trick as with Wald's equation.

Exercise 4 (Martingales)**.** Let $S_n := \sum_{i=1}^n Y_i$ be a symmetric random walk, i.e. the jump sizes $(Y_n, n \geq 1)$ are a sequence of i.i.d. random variables with $\mathbb{P}(Y_n = 1) = 1/2$ and $\mathbb{P}(Y_n = -1) = 1/2$. Let $\mathcal{G}_n := \sigma(Y_1, Y_2, \dots Y_n)$ with $\mathcal{G}_0 := \{\emptyset, \Omega\}$.

(i) Let $\lambda \in \mathbb{R}$. Find a constant $c \in \mathbb{R}$ such that $\exp(\lambda S_n - cn)_{n \in \mathbb{N}}$ is a $(\mathcal{G}_n)$-martingale.

(ii) Prove that $(S_n^2 - n)_{\mathbb{N}_0}$ is a $(\mathcal{G}_n)$-martingale.

(iii) Prove that $(S_n^3 - 3nS_n)_{\mathbb{N}_0}$ is a $(\mathcal{G}_n)$-martingale.

(iv) Find a polynomial $P(s, n)$ with degree 4 on s and degree 2 on n, such that $(P(S_n, n))_{n \in \mathbb{N}_0}$ is a $(\mathcal{G}_n)$-martingale.

(v) Prove that, in general, for a polynomial $P(x, y)$, the process $(P(S_n, n))_{n \in \mathbb{N}_0}$ is a $(\mathcal{G}_n)$-martingale, if

$$P(s + 1, n + 1) + P(s - 1, n + 1) = 2P(s, n).$$

Exercise 5 (Poisson Process is Poisson)**.** In this exercise we are going to assume that the Poisson process $(X_t)_{t \geq 0}$ is defined as the number of events up to time t, where the waiting time between events is always exponentially distributed, i.e.

$$X_t := \max \left\{ k \in \mathbb{N}_0 : T_k := \sum_{i=1}^k Y_i \leq t \right\}, \qquad Y_i \overset{\text{iid}}{\sim} \mathrm{Exp}(\lambda)$$

Show that X_t is Poisson distributed for every $t \geq 0$.

Hint. The exponential distribution is a special case of the Gamma distribution, i.e. $\mathrm{Exp}(\lambda) = \Gamma(1, \frac{1}{\lambda})$. And since the Gamma distribution is stable under summation of iid Gamma distributed random variables, we get $T_k \sim \Gamma(k, \frac{1}{\lambda})$. Using this fact you can either calculate $\mathbb{P}(X_t \leq k)$ using the cumulative distribution function of the Gamma distribution (which you might know or need to calculate), or you could calculate $\mathbb{P}(X_t = k)$ by conditioning on T_k in this sense

$$\mathbb{P}(A) = \mathbb{E}[\mathbb{1}_A] = \mathbb{E}[\mathbb{E}[\mathbb{1}_A \mid T_k]].$$

This avoids the usage of the cdf of the Gamma distribution.