Wahrscheinlichkeitstheorie 1 | **Universität Mannheim**
FSS 2022 | Prof. Leif Döring, Felix Benning

Sheet 2

For the exercise class on the 28.02.2022.
Hand in your solutions at 13:30 in the exercise on Monday 28.02.2022.

Exercise 1. Calculate the following conditional expectations and determine the conditional distributions.

(i) $\mathbb{E}\left[(Y-X)^+|X\right]$, where X, Y are independent uniform random variables on $[0,1]$.

(ii) $\mathbb{E}\left[X|Y\right]$, where (X,Y) has a joint distribution with density:

$$f(x,y) = 4y(x-y)e^{-(x+y)}\mathbb{1}_{0<y<x}.$$

(iii) $\mathbb{E}\left[X|X+Y\right]$, where X, Y are independent Poisson random variables with parameters λ and μ respectively.

Exercise 2 ((Sub-/Super-)Martingales)**.**

(i) The (sub-/super-)martingale property holds over several time steps, i.e.

$$\mathbb{E}[X_m \mid \mathcal{F}_n] \begin{cases} = X_n & \text{martingale} \\ \leq X_n & \text{supermartingale} \\ \geq X_n & \text{submartingale} \end{cases} \quad \forall m \geq n$$

(ii) Expectation increase/decrease/stay-constant, i.e.

$$\mathbb{E}[X_m] \begin{cases} = \mathbb{E}[X_n] & \text{martingale} \\ \leq \mathbb{E}[X_n] & \text{supermartingale} \\ \geq \mathbb{E}[X_n] & \text{submartingale} \end{cases} \quad \forall m \geq n$$

(iii) X is a supermartingale iff $-X$ is a submartingale.

Exercise 3 (Markov Inequality for Conditional Expectation)**.** Prove that for two random variables X, Y and an increasing $h : [0,\infty) \to [0,\infty)$ we have for any $\epsilon > 0$

$$\mathbb{P}(|X| \geq \epsilon \mid Y) \leq \frac{\mathbb{E}[h(|X|) \mid Y]}{h(\epsilon)} \quad \text{a.s.}$$

Remark. *The proof works the same when one conditions on a sigma algebra instead of a random variable Y.*

Exercise 4. Let X, Y be independent with standard normal distribution $\mathcal{N}(0,1)$. Show that, for any $z \in \mathbb{R}$, the conditional distribution $\mathbb{P}(X \in \cdot \mid X+Y = z)$ is $\mathcal{N}(z/2, 1/2)$.

Exercise 5 (Stopping Times)**.**

(i) Suppose $T_1, T_2, \ldots$ are $(\mathcal{F}_n)$-stopping times. Prove that $\inf_{k\in\mathbb{N}} T_k$, $\sup_{k\in\mathbb{N}} T_k$, $\liminf_{k\to\infty} T_k$ and $\limsup_{k\to\infty} T_k$ are stopping times.

(ii) Suppose T, S are $\mathcal{F}_n$ stopping times. Prove $\{S \leq T\} \in \mathcal{F}_{S\wedge T}$ and $\{S = T\} \in \mathcal{F}_{S\wedge T}$.

(iii) (Last Hitting Time is no Stopping Time in General) Consider the stochastic process X defined by $X_0 = 0$, $X_1 \sim \mathcal{U}(\{0,1\})$ and $X_k = 1$ for all $k > 1$. Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ be the natural filtration generated by X. Show that the last hitting time of 0

$$L := \sup\{k \geq 0 : X_k = 0\}$$

is not a stopping time.