Wahrscheinlichkeitstheorie 1 | **Universität Mannheim**
FSS 2022 | Prof. Leif Döring, Felix Benning

Sheet 1

For the exercise class on the 21.02.2022.
Hand in your solutions at 10:15 in the lecture on Monday 21.02.2022.

Exercise 1 (Properties of Conditional Expectation)**.** Let X and Y be two integrable random variables, $\mathcal{G} \subseteq \mathcal{F}$ σ-Algebras.

(i) Prove $\mathbb{E}[\mathbb{E}[X|\mathcal{F}]] = \mathbb{E}[X]$ and $\mathbb{E}[1|\mathcal{F}] = 1$ a.s.

(ii) Prove $\mathbb{E}[\mathbb{E}[X|\mathcal{F}]|\mathcal{G}] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{F}] = \mathbb{E}[X|\mathcal{G}]$ a.s.

(iii) Assume that $X = Y$ almost surely. Prove that, almost surely, $\mathbb{E}[X|Y] = Y$ and $\mathbb{E}[Y|X] = X$.

Exercise 2. The following questions are independent.

(i) Let X and Y be i.i.d. Bernoulli variables: $\mathbb{P}(X = 1) = 1 - \mathbb{P}(X = 0) = p$ for some $p \in [0, 1]$. We set $Z := \mathbf{1}_{\{X+Y=0\}}$. Compute $\mathbb{E}[X|Z]$ and $\mathbb{E}[Y|Z]$. Are these random variables independent ?

(ii) Let X be a square-integrable random variable and $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra. We define the *conditional variance*
$$\mathrm{Var}(X|\mathcal{G}) := \mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])^2|\mathcal{G}].$$
Prove the following identity:
$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|\mathcal{G})] + \mathrm{Var}(\mathbb{E}[X|\mathcal{G}]).$$

Exercise 3 (Factorization lemma)**.** (Important!) Let $X, Y : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be random variables. Show that, if X is $(\sigma(Y), \mathcal{B}(\mathbb{R}))$-measurable, then there exists a measurable function $g : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, such that $X = g(Y)$.

Exercise 4 (Best Estimators)**.** Let X be a real random variable.

(i) Find the estimator m minimizing $\mathbb{E}[\mathbb{1}_{X \neq m}]$ when X is discrete. What is the best estimator for a dice roll using this loss?

We define the median of X to be $\mathbb{M}[X] := \inf\{m \in \mathbb{R} : f(m) > 0\}$, where
$$f : \begin{cases} \mathbb{R} \to [-1, 1] \\ m \mapsto \mathbb{P}(X \leq m) - \mathbb{P}(X > m) \end{cases}$$

(ii) (Unimportant) Prove that $\mathbb{P}(X \leq \mathbb{M}[X]) \geq \frac{1}{2}$ and $\mathbb{P}(X \geq \mathbb{M}[X]) \geq \frac{1}{2}$, and show that for all $m > \mathbb{M}[X]$ we do not have $\mathbb{P}(X \leq m) \geq \frac{1}{2}$.

Hint. Calculate the limits $\lim_{m \uparrow \mathbb{M}[X]} f(m)$ and $\lim_{m \downarrow \mathbb{M}[X]} f(m)$ using the continuity of measures. Also note that $f(m) = 2F_X(m) - 1$, where F_X is the cumulative distribution function of X.

Remark. *This property is usually the defining property of the median. If this property does not define the median uniquely,* $\mathbb{M}[X]$ *is the largest median. A lower bound can be found similarly.*

(iii) Prove that the median minimizes the L^1 error, i.e. $\mathbb{M}[X] = \arg\min_m \mathbb{E}[|X - m|]$. Whenever useful, you can assume continuity.

Hint. Recall that for $X \geq 0$ a.s., we have $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X \geq t)dt = \int_0^\infty \mathbb{P}(X > t)dt$. Use this fact to prove

$$\mathbb{E}[(X - m)_+] = \int_m^\infty \mathbb{P}(X > m)dt.$$