Sheet 9

Hand in your solutions before 17:00 Thursday 23/April/2020.

Exercise 1 (Convergence of Gaussian random variables)**.** Let $(X_n, n \geq 1)$ be a sequence of Gaussian random variables with $X_n \sim \mathcal{N}(m_n, \sigma_n^2)$ (i.e. with mean value m_n and variance σ_n^2), defined on the same probability space. Suppose that X_n converges to a random variable X in L^2 as $n \to \infty$.

(i) Prove that, the two limits $m := \lim_{n\to\infty} m_n$ and $\sigma^2 := \lim_{n\to\infty} \sigma_n^2$ exist. Moreover, the limit X is a Gaussian random variable $X \sim \mathcal{N}(m, \sigma^2)$.

(ii) Prove that, $\lim_{n\to\infty} X_n = X$ also holds in L^p with any $1 \leq p < \infty$.

Exercise 2 (Symmetric random walk; Gambler's ruin revisited)**.** Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$. Let $a, b \in \mathbb{N}$ with $0 < a < b$. Define $S_0 \coloneqq a$ and for every $n \geq 1$, $S_n \coloneqq S_{n-1} + X_n$. Finally, define the following stopping time:

$$T_0 \coloneqq \inf\{n \geq 0 : S_n = 0\} \text{ and } T_b \coloneqq \inf\{n \geq 0 : S_n = b\}.$$

Let $T = T_0 \wedge T_b$.

(i) Deduce the values of $\mathbb{P}(T_0 < T_b)$ and $\mathbb{P}(T_0 > T_b)$.

(ii) Compute the value of $\mathbb{E}[T]$.

(iii) Show that $\mathbb{P}(T_0 < \infty) = 1$ and $\mathbb{P}(T_b < \infty) = 1$.

(iv) I have 20 euros initially and I play in a fair game. I have 2 different choices: either I always bet for 10 euros every round; or I always bet for 1 euro each round. I will leave the game whenever I have 100 euros or I lose all my money. Which one should I choose, in order to reduce my risk to lose all? What if the game is favourable or unfavourable?

Hint: you have many martingales for use; see Ex2 in Sheet5. Be careful with the assumptions in optional stopping theorem.

Exercise 3 (Square integrable martingale)**.** Let $(X_n, n \in \mathbb{N}_0)$ be a martingale, with $X_0 = 0$ and $\mathbb{E}\left[X_n^2\right] < \infty, \forall n \in \mathbb{N}$. (attention, but we don NOT assume $(X_n, n \in \mathbb{N}_0)$ is bounded in L^2)

(i) Prove that X_n^2 is a submartingale; and give the following Doob's decomposition:

$$X_n^2 = M_n + A_n,$$

where

$$A_n = \sum_{m=1}^{n} \mathbb{E}\left[(X_m - X_{m-1})^2 | \mathcal{F}_{m-1}\right]$$

and $M_n = X_n^2 - A_n$. Prove that M_n is a martingale, and A_n is a predictable increasing process. A_n is often denoted by $\langle X, X\rangle_n$.

(ii) Set $\tau_a := \inf\{n : A_{n+1} > a^2\}$. Prove that τ_a is a stopping time.

(iii) Let $A_\infty := \lim_{n\to\infty} A_n$, so A_∞ is a random variable with its value in $\mathbb{R}_+ \cup \{+\infty\}$. Prove that

$$\mathbb{E}\left[\sup_{n\geq 0} |X_n|^2\right] \leq 4\mathbb{E}\left[A_\infty\right].$$

(iv) Deduce that $\lim_{n\to\infty} X_n$ exists and is finite a.s. on $\{A_\infty < \infty\}$. That is to say, there exists an event E with $\mathbb{P}(E) = 1$, such that for every $\omega \in E \cap \{A_\infty < \infty\}$, $\lim_{n\to\infty} X_n(\omega)$ exists and is finite.
Hint: You may first study the martingale $X_{n\wedge\tau_a}$.

(v) Show that $\mathbb{P}\left(\sup_{k\leq n} |X_{k\wedge\tau_a}| > a\right) \leq a^{-2}\mathbb{E}\left[a^2 \wedge A_\infty\right]$.

(vi) Prove that $\mathbb{P}\left(\sup_{n\geq 0} |X_n| > a\right) \leq \mathbb{P}\left(A_\infty > a^2\right) + \mathbb{P}\left(\sup_{n\geq 0} |X_{n\wedge\tau_a}| > a\right)$.

(vii) Prove that $\mathbb{E}\left[\sup_{n\geq 0} |X_n|\right] \leq 3\mathbb{E}\left[A_\infty^{1/2}\right]$.

(viii) Let ξ_i be a sequence of i.i.d. random variables with $\mathbb{E}\left[\xi_i\right] = 0$, and $\mathbb{E}\left[\xi_i^2\right] < \infty$. Let $S_0 := 0$, $S_n := \sum_{i=1}^n \xi_i$, then S_n is a square integrable martingale. Show that if τ is a stopping time with $\mathbb{E}\left[\sqrt{\tau}\right] < \infty$, then $\mathbb{E}\left[S_\tau\right] = 0$.

Exercise 4. Let $C := C([0,\infty), \mathbb{R})$, the space of all real-valued continuous functions on $[0,1]$. The space C is endowed with the d: for any $f, g \in C$,

$$d(f,g) := \sum_{n=1}^{\infty} 2^{-n} \left(\left(\sup_{s\in[0,n]} |f(s) - g(s)|\right) \wedge 1\right).$$

Let $\mathcal{B}$ be the Borel σ-algebra of the metric space (C, d): we accept that $\mathcal{B}$ is generated by the collection of all open balls:

$$\mathcal{B} := \sigma\left(B(f,r) \colon f \in C, r > 0\right), \quad \text{where } B(f,r) := \{g \colon g \in C, d(f,g) < r\}.$$

For every $x \geq 0$, define a function $\phi_t \colon C \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by

$$\phi_t(f) = f(t), \qquad f \in C.$$

Let $\mathcal{C} := \sigma(\phi_t, t \geq 0)$ be the minimum σ-algebra such that all functions $\phi_t, t \geq 0$ are measurable.

(i) Fix any $f \in C$ and $n \in \mathbb{N}$. Define a function $d_f^{(n)} \colon C \to \mathbb{R}$ by

$$d_f^{(n)}(g) := \sup_{s\in[0,n]} |f(s) - g(s)|, \quad \forall g \in C.$$

Show that $d_f^{(n)}$ is $(\mathcal{C}, \mathcal{B}(\mathbb{R}))$-measurable.

(ii) Fix any $f \in C$. Define a function $d_f \colon C \to \mathbb{R}$ by $d_f(g) = d(f,g)$, $\forall g \in C$. Show that d_f is $(\mathcal{C}, \mathcal{B}(\mathbb{R}))$-measurable.

(iii) Prove the identity $\mathcal{B} = \mathcal{C}$.