Wahrscheinlichkeitstheorie 2 **Universität Mannheim**
FSS 2020 Dr. Q. Shi

Sheet 8

Hand in your solutions before 17:00 Thursday 02/April/2020.

Exercise 1. Show that the following families are uniformly integrable:

(i) a finite family $\{X_1, X_2, \dots X_n\}$ with each $\mathbb{E}[|X_i|] < \infty$.

(ii) a sequence of identically distributed random variables $\{X_i, i \in \mathbb{N}\}$ with $\mathbb{E}[|X_1|] < \infty$.

(iii) Let $(X_i, i \in \mathbb{N})$ be a sequence of i.i.d. random variables with $\mathbb{E}[|X_1|] < \infty$. Let $S_n = \sum_{i=1}^n X_i$ for every $n \in \mathbb{N}$. Then the family $\{\frac{1}{n} S_n, n \in \mathbb{N}\}$ is uniformly integrable. **Hint:** you may use (ii).

Exercise 2. Let $(Z_i, \imath \geq 1)$ be i.i.d. centred random variables with $\mathbb{E}[|Z_i|] < \infty$. Let θ be a random variable with $\mathbb{E}[|\theta|] < \infty$. They are all defined in the same probability space. Let $Y_i := Z_i + \theta$ for $i \geq 1$. In statistical terms, we may understand (Y_i) as a sample (observations/data) of θ with error (Z_i) (typically, Z_i can be iid standard normal distribution). The distribution of θ is called the *prior distribution* and the conditional distribution of θ given $(Y_1, \dots Y_n)$ is called the *posterior distribution* after n observations. Let $\mathcal{F}_\infty = \sigma(Y_1, Y_2, Y_3, \dots)$. Show that $\lim_{n\to\infty} \mathbb{E}[\theta \mid Y_1, \dots Y_n] = \theta$ a.s..
Hint: using SLLN to prove $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n Y_i$ is $\mathcal{F}_\infty$-measurable. .

Exercise 3 (Product measurable space)**.** Let $(\Omega_\alpha, \mathcal{F}_\alpha)_{\alpha\in I}$ be a family of measurable spaces. For any collection $A \subset I$, we define a space Ω_A by the Cartesian product

$$\Omega_A := \prod_{\alpha\in A} \Omega_\alpha;$$

i.e. each element in Ω_A is a tuple $(x_\alpha)_{\alpha\in A}$ with $x_\alpha \in \Omega_\alpha$. For any $B \subset A$, we define the projection map $\pi_{A\to B} \colon \Omega_A \to \Omega_B$ by

$$\pi_{A\to B}((x_\alpha)_{\alpha\in A}) = (x_\alpha)_{\alpha\in B}.$$

For simplicity, we denote $\pi_A := \pi_{I\to A}$ and $\pi_\alpha := \pi_{\{\alpha\}}$. Moreover, we define a *product σ-algebra $\mathcal{F}_A$* on Ω_A by

$$\mathcal{F}_A := \prod_{\alpha\in A} \mathcal{F}_\alpha := \sigma\left(\pi^{-1}_{A\to\{\alpha\}}(B) \colon B \in \mathcal{F}_\alpha, \alpha \in A\right).$$

Recall that $\mathcal{F}_A$ is the smallest σ-algebra on Ω_A such that $(\pi_{A\to\{\alpha\}}, \alpha \in A)$ are all measurable, i.e. $\mathcal{F}_A = \sigma(\pi_{A\to\{\alpha\}}, \alpha \in A)$.

(i) For any $B \subset A \subset I$, show that $\pi_{A\to B}$ is $(\mathcal{F}_A, \mathcal{F}_B)$-measurable.

(ii) For any $E_I \in \mathcal{F}_I$ show that there exists an at most countable set $B \subset I$ and $E_B \in \mathcal{F}_B$, such that $E_I = \pi_B^{-1}(E_B)$. **Hint**: consider the family of sets in $\mathcal{F}_I$ that satisfy this property and show that this family is a σ-algebra.

(iii) If I is at most countable, show that there is the identity

$$\mathcal{F}_I = \sigma\left(\prod_{\alpha\in I} B_\alpha, B_\alpha \in \mathcal{F}_\alpha\right).$$

Is this still true when I is uncountable?