Wahrscheinlichkeitheorie 2 | **Universität Mannheim**
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Sheet 7

We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_n)_{n\in\mathbb{N}_0}, \mathbb{P})$ for all the exercises. All the random variables are assumed to be real-valued.

Exercise 1 (Kolmogorov 0-1 law)**.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables. Define a filtration by $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$, for every $n \in \mathbb{N}$. We also denote $\mathcal{F}^n := \sigma(X_{n+1}, X_{n+2}, \dots)$ (but $(\mathcal{F}^n)$ is not a filtration). Let $\mathcal{F}^\infty := \bigcap_{n\in\mathbb{N}} \mathcal{F}^n$. Let $A \in \mathcal{F}^\infty$. By using the martingale $\mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_n]$, prove that, $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

Exercise 2. Let $(X_n)_{n\geq 1}$ be a sequence of independent, non-negative random variables of mean 1. Consider $(\mathcal{F}_n)_{n\geq 0}$ its natural filtration. Let $M_0 := 1$ and for every $n \geq 1$,

$$M_n := \prod_{k=1}^{n} X_k.$$

(i) Prove that $M := (M_n)_{n\geq 0}$ is a martingale which converges almost surely.

(ii) Denote by $M_\infty := \lim_{n\to\infty} M_n$. Prove that the following are equivalent:

- (a) $\mathbb{E}[M_\infty] = 1$.
- (b) $\lim_{n\to\infty} M_n = M_\infty$ in L^1.
- (c) $(M_n)_{n\geq 0}$ is uniformly integrable.
- (d) $\prod_{k=1}^{\infty} a_n > 0$ where $0 < a_n := \mathbb{E}[\sqrt{X_n}] \leq 1$.

Moreover, if one (and then every one) of the above statements fails to hold, then $M_\infty = 0$ almost surely.
Hint: you can prove (d) $\Longrightarrow$ (c) by considering the process defined for every $n \geq 1$ by

$$N_n := \prod_{k=1}^{n} \frac{\sqrt{X_k}}{a_k},$$

which is a martingale bounded in L^2 (to prove). Then use it to prove that M is dominated by an integrable random variable.

Remark: By a result in analysis, we also know that (d) is equivalent to

(e) $\sum_{k=1}^{\infty}(1 - a_n) < \infty$.

Exercise 3. Extension of the Borel-Cantelli Lemmas
Let $(E_n)_{n\geq 1}$ be a sequence of events, with $E_n \in \mathcal{F}_n$ for every $n \geq 1$. Define $X_0 = Y_0 = M_0 = 0$ and for every $n \geq 1$

$$X_n := \sum_{k=1}^{n} \mathbf{1}_{E_k}, \quad Y_n := \sum_{k=1}^{n} \mathbb{P}(E_k | \mathcal{F}_{k-1}) \quad \text{and} \quad M_n := X_n - Y_n.$$

Let $X_\infty := \lim_{n\to\infty} X_n$ and $Y_\infty := \lim_{n\to\infty} Y_n$. We aim to show that

$$\{Y_\infty < \infty\} = \{X_\infty < \infty\} \qquad \text{almost surely.}$$

(i) Explain why this strengthens the two Borel-Cantelli lemmas.

(ii) Prove that $(X_n)_{n\geq 0}$ is a submartingale and $(M_n)_{n\geq 0}$ is a martingale.

(iii) Show that for every $a > 0$, $\tau_a := \inf\{n \geq 0 : Y_{n+1} > a\}$ is a stopping time.

(iv) Show that for every $a > 0$, $n \geq 0$, $M^-_{n\wedge\tau_a} \leq a$. Deduce that $(M_{n\wedge\tau_a})_{n\geq 0}$ converges almost surely.

(v) Show that $\{Y_\infty < \infty\} \subset \{X_\infty < \infty\}$ almost surely. **Hint:** What happens on $\{\tau_a = \infty\}$?

(vi) Show that $\{X_\infty < \infty\} \subset \{Y_\infty < \infty\}$ almost surely. **Hint:** Consider $\sigma_a := \inf\{n \geq 0 : X_n > a\}$ and $M^+_{n\wedge\sigma_a}$.

Exercise 4. Let $\mathcal{F}_\infty := \sigma(\bigcup_{k=1}^\infty \mathcal{F}_k)$. Assume that there exists a finite measure ν on $\mathcal{F}_\infty$, such that, for all $n \geq 0$, there exists a non-negative, $\mathcal{F}_n$-measurable and $\mathbb{P}$-integrable function M_n with

$$\nu(A) = \mathbb{E}[M_n \mathbf{1}_A], \quad \text{for every } A \in \mathcal{F}_n.$$

In particular, we have that $\nu \ll \mathbb{P}$ (ν is absolutely continuous with respect to $\mathbb{P}$) on $\mathcal{F}_n$ for all $n \geq 0$.

(i) Prove that $(M_n)_{n\geq 0}$ is a martingale which converges almost surely. Denote by M_∞ the limit.

(ii) Prove that $\nu \ll \mathbb{P}$ on $\mathcal{F}_\infty$ if and only if $(M_n)_{n\geq 0}$ is closed, and that in that case,

$$\nu(A) = \mathbb{E}[M_\infty \mathbf{1}_A], \quad \text{for every } A \in \mathcal{F}_\infty.$$

(iii) Assume that ν and $\mathbb{P}$ are orthogonal on $\mathcal{F}_\infty$, i.e. there exists $S \in \mathcal{F}_\infty$ such that $\mathbb{P}(S) = 1$ and $\nu(S) = 0$. Prove that $M_\infty = 0$ almost surely.