Wahrscheinlichkeitheorie 2 | **Universität Mannheim**
FSS 2020 | Prof. M. Slowik, Dr. Q. Shi

Sheet 6

We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$ for all the exercises. All the random variables are assumed to be real-valued.

Exercise 1 (Versions of optional stopping theorem)**.** Let $(X_n)_{n\in\mathbb{N}_0}$ be a $\mathbb{F}$-submartingale. Let T be $\mathbb{F}$-stopping time.

(i) Suppose that $(X_n)_{n\in\mathbb{N}_0}$ is bounded and T is a.s. finite. Prove that $\mathbb{E}[X_0] \leq \mathbb{E}[X_T] < \infty$.

(ii) Suppose that there exists a constant $K > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$|X_n(\omega) - X_{n-1}(\omega)| \leq K, \forall n \in \mathbb{N}.$$

We also suppose that $\mathbb{E}[T] < \infty$. Show that $\mathbb{E}[X_0] \leq \mathbb{E}[X_T] < \infty$.

Exercise 2. Let $(X_n)_{n\in\mathbb{N}_0}$ be a $\mathbb{F}$-submartingale. Let S, T be bounded $\mathbb{F}$-stopping times with $S \leq T$.

(i) Prove that $\mathbb{E}[X_S] \leq \mathbb{E}[X_T] < \infty$.
Hint: consider $H_n := \mathbf{1}_{\{S<n\leq T\}} = \mathbf{1}_{\{S\leq n-1\}} - \mathbf{1}_{\{T\leq n-1\}}$.

(ii) Recall that $\mathcal{F}_S$ is the σ-algebra associated with the stopping time S. Prove that

$$\mathbb{E}[X_T \mid \mathcal{F}_S] \geq X_S.$$

Hint: consider $\tau := S\mathbf{1}_A + T\mathbf{1}_{A^c}$ for each $A \in \mathcal{F}_S$.

Exercise 3. Let $(X_n)_{n\in\mathbb{N}_0}$ be an integrable and $\mathbb{F}$-adapted process. Suppose that

$$\mathbb{E}[X_T] = \mathbb{E}[X_0], \quad \text{for every bounded } \mathbb{F}\text{-stopping time } T.$$

Show that $(X_n)_{n\in\mathbb{N}_0}$ is a $\mathbb{F}$-martingale.
Hint: It suffices to prove that for every $n \in \mathbb{N}_0$, for every $A \in \mathcal{F}_n$, $\mathbb{E}[X_{n+1}\mathbf{1}_A] = \mathbb{E}[X_n\mathbf{1}_A]$, or equivalently, $\mathbb{E}[X_{n+1}\mathbf{1}_A + X_n\mathbf{1}_{A^c}] = \mathbb{E}[X_n]$.

Exercise 4. ABRACADABRA
At each of times $1, 2, \ldots$, a monkey types a capital letter at random, the sequence of letters typed forming an i.i.d. sequence of random variables chosen uniformly amongst the 26 possible capital letters. Let T be the first time by which the monkey has produced the consecutive sequence ABRACADABRA. We prove that

$$\mathbb{E}[T] = 26^{11} + 26^4 + 26.$$

For this, imagine that just before **each** time $n \in \mathbb{N}$, a new gambler indexed by n arrives on the scene. He bets 1 euro that

the n-th letter will be A.

If he loses, he loses all his money and leaves. If he wins, he receives 26 euros, all of which he bets on the event that

the $n+1$-th letter will be B.

If he loses, he loses all his money and leaves. If he wins, he bets his whole current fortune of 26^2 that

the $n+2$-th letter will be R.

and so on through the ABRACADABRA sequence. All gamblers come and play and leave independently of each other. Let $(X_i^k, i \geq 0)$ denote the fortune of the k-th gambler just after time i: in particular, $X_i^k = 1$ for all $i \leq k-1$.

(i) Show that indeed, $\mathbb{E}[T] < \infty$. **Hint:** See exercise 2 in Sheet 4.

(ii) For every $n \geq 1$, let M_n be the total net profit of all gamblers who have participated in the game just after time n, i.e.

$$M_n = \sum_{k=1}^{n} (X_n^k - 1),$$

and $M_0 := 0$.

Show that $(M_n)_{n\geq 0}$ is a martingale.

(iii) Show that $T = \inf\{n \geq 0 : M_n + n \geq 26^{11} + 26^4 + 26\}$. Conclude that $\mathbb{E}[T] = 26^{11} + 26^4 + 26$.