

Sheet 5

For the exercise class 16/03/2020. Hand in your solutions before 17:00 Thursday 12/03/2020.

We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_n)_{n \geq 0}, \mathbb{P})$ for all the exercises. All the random variables are assumed to be real-valued.

Exercise 1. Let $(H_n)_{n \in \mathbb{N}_0}$ be a *predictable* process respect to the filtration \mathbb{F} , i.e. H_0 is a constant and for every $n \in \mathbb{N}$, H_n is \mathcal{F}_{n-1} measurable. Show that, if the \mathbb{F} -predictable process (H_n) is also a \mathbb{F} -martingale, then $H_n = H_0$ a.s. for all $n \in \mathbb{N}$.

Exercise 2. Let $(X_n, n \geq 1)$ is a sequence of i.i.d. random variables with $\mathbb{P}(X_n = 1) = 1/2$ and $\mathbb{P}(X_n = -1) = 1/2$. Let $\mathcal{G}_n := \sigma(X_1, X_2, \dots, X_n)$ with $\mathcal{G}_0 := \{\emptyset, \Omega\}$. Let $S_n := \sum_{i=1}^n X_i$ for every $n \in \mathbb{N}_0$.

(i) Prove that $(S_n^2 - n)_{n \in \mathbb{N}_0}$ is a (\mathcal{G}_n) -martingale.

(ii) Prove that $(S_n^3 - 3nS_n)_{n \in \mathbb{N}_0}$ is a (\mathcal{G}_n) -martingale.

(iii) Find a polynomial $P(s, n)$ with degree 4 on s and degree 2 on n , such that $(P(S_n, n))_{n \in \mathbb{N}_0}$ is a (\mathcal{G}_n) -martingale.

(iv) Prove that, in general, for a polynomial $P(x, y)$, the process $(P(S_n, n))_{n \in \mathbb{N}_0}$ is a (\mathcal{G}_n) -martingale, if

$$P(s+1, n+1) + P(s-1, n+1) = 2P(s, n).$$

(v) Let $\lambda \in \mathbb{R}$. Find a constant $c \in \mathbb{R}$ such that $\exp(\lambda S_n - cn)_{n \in \mathbb{N}}$ is a (\mathcal{G}_n) -martingale.

Exercise 3. De Mere's martingale

Consider a fair game of heads and tails: you may bet for k euros, then with probability $1/2$ you win and is rewarded $2k$ euros and otherwise you lose the k euros). A player adopts the following strategy. He bets 1 euro at the first hand. If this first bet is lost he then bets 2 euros at the second hand. If he loses his n first bets, he bets 2^n euros at the $(n+1)$ -th hand. Moreover, as soon as the player wins one bet, he stops playing (or equivalently, after he wins a bet, he bets 0 euro on every sub-sequential hand). Denote by $(X_n)_{n \geq 1}$ the net profit of the player just after the n -th hand has been played.

(i) Show that $(X_n)_{n \geq 1}$ is a martingale.

(ii) Show that almost surely, the game ends in finite time. What is the expectation of the duration of the game? What is the net profit of the player at the moment when he stops playing?

(iii) Compute the expectation of his maximal loss during the game.

(iv) What would change if the player decides to triple his bet after every hand that he has lost? What changes if the game is unfair? Why do you think casinos have (in effect) forbidden this strategy by limiting the maximum possible bet?

Exercise 4. Gambler's ruin

Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{P}(X_1 = 1) = 1 - \mathbb{P}(X_1 = -1) = p$ for some $p \in (0, 1)$, $p \neq 1/2$. Let $a, b \in \mathbb{N}$ with $0 < a < b$. Define $S_0 := a$ and for every $n \geq 1$, $S_n := S_{n-1} + X_n$. Finally, define the following stopping time:

$$T := \inf\{n \geq 0 : S_n = 0 \text{ or } S_n = b\}.$$

We consider the filtration generated by $(X_n)_{n \geq 1}$.

- (i) Show that $\mathbb{E}[T] < \infty$. **Hint:** Sheet 4 provides a sufficient condition.
- (ii) Consider for every $n \geq 0$,

$$M_n := \left(\frac{1-p}{p}\right)^{S_n} \quad \text{and} \quad N_n := S_n - n(2p-1).$$

Prove that $(M_n)_{n \geq 0}$ and $(N_n)_{n \geq 0}$ are martingales.

- (iii) Deduce the values of $\mathbb{P}(S_T = 0)$ and $\mathbb{P}(S_T = b)$.
- (iv) Compute the value of $\mathbb{E}[T]$.