Wahrscheinlichkeitheorie 2 — **Universität Mannheim**

FSS 2020 — Prof. M. Slowik, Dr. Q. Shi

Sheet 5

For the exercise class 16/03/2020. Hand in your solutions before 17:00 Thursday 12/03/2020.

We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$ for all the exercises. All the random variables are assumed to be real-valued.

Exercise 1. Let $(H_n)_{n\in\mathbb{N}_0}$ be a *predictable* process respect to the filtration $\mathbb{F}$, i.e. H_0 is a constant and for every $n \in \mathbb{N}$, H_n is $\mathcal{F}_{n-1}$ measurable. Show that, if the $\mathbb{F}$-predictable process (H_n) is also a $\mathbb{F}$-martingale, then $H_n = H_0$ a.s. for all $n \in \mathbb{N}$.

Solution. H_n is a martingale:

$$\mathbb{E}[H_n \mid \mathcal{F}_{n-1}] = H_{n-1}.$$

H_n is predictable:

$$\mathbb{E}[H_n \mid \mathcal{F}_{n-1}] = H_n.$$

So we have $H_n = H_{n-1}$ a.s. for each $n \in \mathbb{N}$. Therefore, $H_n = H_0$ a.s. for all $n \in \mathbb{N}$. □

Exercise 2. Let $(X_n, n \geq 1)$ is a sequence of i.i.d. random variables with $\mathbb{P}(X_n = 1) = 1/2$ and $\mathbb{P}(X_n = -1) = 1/2$. Let $\mathcal{G}_n := \sigma(X_1, X_2, \dots X_n)$ with $\mathcal{G}_0 := \{\emptyset, \Omega\}$. Let $S_n := \sum_{i=1}^n X_i$ for every $n \in \mathbb{N}_0$.

(i) Prove that $(S_n^2 - n)_{\mathbb{N}_0}$ is a $(\mathcal{G}_n)$-martingale.

(ii) Prove that $(S_n^3 - 3nS_n)_{\mathbb{N}_0}$ is a $(\mathcal{G}_n)$-martingale.

(iii) Find a polynomial $P(s, n)$ with degree 4 on s and degree 2 on n, such that $(P(S_n, n))_{n\in\mathbb{N}_0}$ is a $(\mathcal{G}_n)$-martingale.

(iv) Prove that, in general, for a polynomial $P(x, y)$, the process $(P(S_n, n))_{n\in\mathbb{N}_0}$ is a $(\mathcal{G}_n)$-martingale, if

$$P(s + 1, n + 1) + P(s - 1, n + 1) = 2P(s, n).$$

(v) Let $\lambda \in \mathbb{R}$. Find a constant $c \in \mathbb{R}$ such that $\exp(\lambda S_n - cn)_{n\in\mathbb{N}}$ is a $(\mathcal{G}_n)$-martingale.

Solution. We first prove (iv). Since S_n is $\mathcal{G}_n$-measurable and $|S_n| \leq n$, we know that for each polynomial P, the random variable $P(S_n, n)$ is bounded and $\mathcal{G}_n$-measurable. Moreover, we have

$$\begin{aligned}
\mathbb{E}[P(S_{n+1}, n+1) \mid \mathcal{G}_n] &= \mathbb{E}[P(S_n + X_{n+1}, n+1)(\mathbf{1}_{\{X_{n+1}=1\}} + \mathbf{1}_{\{X_{n+1}=-1\}}) \mid \mathcal{G}_n] \\
&= \mathbb{E}[P(S_n + 1, n+1)\mathbf{1}_{\{X_{n+1}=1\}} \mid \mathcal{G}_n] + \mathbb{E}[P(S_n - 1, n+1)\mathbf{1}_{\{X_{n+1}=-1\}} \mid \mathcal{G}_n] \\
&= P(S_n + 1, n+1)\mathbb{P}(X_{n+1} = 1) + P(S_n - 1, n+1)\mathbb{P}(X_{n+1} = -1) \\
&= \frac{1}{2}\left(P(S_n + 1, n+1) + P(S_n - 1, n+1)\right),
\end{aligned}$$

where we have used the fact that $P(S_n + 1, n + 1)$ is $\mathcal{G}_n$-measurable for the third equality. If

$$P(s + 1, n + 1) + P(s - 1, n + 1) = 2P(s, n),$$

then we deduce that

$$\mathbb{E}[P(S_{n+1}, n+1) \mid \mathcal{G}_n] = P(S_n, n).$$

So $(P(S_n, n), n \in \mathbb{N})$ is a $\mathcal{G}_n$-martingale.

Using (iv), one can deduce (i) (ii) and (iii) by straightforward calculation.

(v): Since $|S_n| \leq n$, $\exp(\lambda S_n - cn)$ is bounded and thus integrable.

$$\begin{aligned}\mathbb{E}[\exp(\lambda S_n - cn) \mid \mathcal{F}_{n-1}] &= \mathbb{E}[\exp(\lambda S_{n-1} - cn)\exp(\lambda X_n) \mid \mathcal{F}_{n-1}] \\ &= \exp(\lambda S_{n-1} - cn)\mathbb{E}[\exp(\lambda X_n)] \\ &= \exp(\lambda S_{n-1} - c(n-1))e^{-c}\cosh(\lambda).\end{aligned}$$

So this is a martingale if $c = \log(\cosh(\lambda))$. □

Exercise 3. De Mere's martingale

Consider a fair game of heads and tails: you may bet for k euros, then with probability $1/2$ you win and is rewarded $2k$ euros and otherwise you lose the k euros). A player adopts the following strategy. He bets 1 euro at the first hand. If this first bet is lost he then bets 2 euros at the second hand. If he loses his n first bets, he bets 2^n euros at the $(n+1)$-th hand. Moreover, as soon as the player wins one bet, he stops playing (or equivalently, after he wins a bet, he bets 0 euro on every sub-sequential hand). Denote by $(X_n)_{n\geq 1}$ the net profit of the player just after the n-th hand has been played.

(i) Show that $(X_n)_{n\geq 1}$ is a martingale.

(ii) Show that almost surely, the game ends in finite time. What is the expectation of the duration of the game? What is the net profit of the player at the moment when he stops playing?

(iii) Compute the expectation of his maximal loss during the game.

(iv) What would change if the player decides to triple his bet after every hand that he has lost? What changes if the game is unfair? Why do you think casinos have (in effect) forbidden this strategy by limiting the maximum possible bet?

Solution.

(i) Let X'_n denote the net profit of the player just after the n-th hand has been played, if you never stop. $X'_1 \in \{-1, 1\}$ almost surely, and for any $n \geq 1$, $X'_{n+1} \in \{X'_n - 2^n, X'_n + 2^n\}$ almost surely. Thus for every $n \geq 1$, X'_n is integrable. Moreover, for every $n \geq 1$,

$$\mathbb{P}(X'_{n+1} = X'_n - 2^n | X'_n) = \mathbb{P}(X'_{n+1} = X'_n + 2^n | X'_n)$$

so $\mathbb{E}[X'_{n+1} - X'_n | X'_n] = 0$. That is, $(X'_n)_{n\geq 1}$ is a martingale.

Let T be the first time that he wins. We can write

$$T = \inf\{n \geq 1 \colon X'_n > X'_{n-1}\}.$$

This is a stopping time. Then by the optional stopping theorem, the process $(X_n = X'_{n\wedge T})_{n\geq 1}$ is a martingale.

(ii) The game ends at the first win. The distribution of T is Geometric of parameter $1/2$ which is almost surely finite, and $\mathbb{E}[T] = 2$. Finally, almost surely,

$$X_T = \sum_{n=1}^{\infty} X_n \mathbf{1}_{\{T=n\}} = \sum_{n=1}^{\infty} \Big(- \sum_{k=1}^{n} 2^{k-1} + 2^n \Big) \mathbf{1}_{\{T=n\}} = \sum_{n=1}^{\infty} \mathbf{1}_{\{T=n\}} = 1.$$

(iii) The maximal loss is given by X_{T-1} and, almost surely,

$$X_{T-1} = \sum_{n=1}^{\infty} X_{n-1} \mathbf{1}_{\{T=n\}} = \sum_{n=1}^{\infty} - \sum_{k=1}^{n-1} 2^{k-1} \mathbf{1}_{\{T=n\}} = \sum_{n=1}^{\infty} (1 - 2^{n-1}) \mathbf{1}_{\{T=n\}} = 1 - 2^{T-1}.$$

Then,

$$\mathbb{E}[X_{T-1}] = 1 - \mathbb{E}[2^{T-1}] = 1 - \sum_{n=1}^{\infty} 2^{n-1} \frac{1}{2^n} = 1 - \sum_{n=1}^{\infty} \frac{1}{2} = -\infty.$$

The strategy is risky in the real world since the initial fortune is finite (possibility of being bankrupt before winning).

(iv) If the player decides to triple his bet after every hand that he has lost, then $X_T = \frac{1}{2}(3^T - 1)$, but $X_{T-1} = 1 - 3^{T-1}$ so, again, $\mathbb{E}[X_{T-1}] = -\infty$.

If the game is unfair: denote by $p \in [0, 1]$ the probability to win each bet. Then $\mathbb{E}[T] = 1/p$ and $X_T = 1$.

$$\mathbb{E}[X_{T-1}] = 1 - \mathbb{E}[2^{T-1}] = 1 - \sum_{n=1}^{\infty} 2^{n-1} (1-p)^{n-1} p = \begin{cases} -\infty & p \leq 1/2, \\ 1 - \frac{p}{1-2(1-p)} & p > 1/2. \end{cases}$$

Why do Casino games have maximum betting limits? You can have your own explain. Here is someone's arguments:
`https://www.casinotestreports.com/casino-maximum-bet-limits`.

□

Exercise 4. Gambler's ruin
Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{P}(X_1 = 1) = 1 - \mathbb{P}(X_1 = -1) = p$ for some $p \in (0, 1)$, $p \neq 1/2$. Let $a, b \in \mathbb{N}$ with $0 < a < b$. Define $S_0 := a$ and for every $n \geq 1$, $S_n := S_{n-1} + X_n$. Finally, define the following stopping time:

$$T := \inf\{n \geq 0 : S_n = 0 \text{ or } S_n = b\}.$$

We consider the filtration generated by $(X_n)_{n\geq 1}$.

(i) Show that $\mathbb{E}[T] < \infty$. **Hint:** Sheet 4 provides a sufficient condition.

(ii) Consider for every $n \geq 0$,

$$M_n := \left(\frac{1-p}{p} \right)^{S_n} \quad \text{and} \quad N_n := S_n - n(2p - 1).$$

Prove that $(M_n)_{n\geq 0}$ and $(N_n)_{n\geq 0}$ are martingales.

(iii) Deduce the values of $\mathbb{P}(S_T = 0)$ and $\mathbb{P}(S_T = b)$.

(iv) Compute the value of $\mathbb{E}[T]$.

Solution.

(i) We have for every $n \geq 0$, $\mathbb{P}(T \leq n + b | \mathcal{F}_n) > \inf(p^b, (1-p)^b) > 0$. We deduce from Sheet 4, Exercise 2 that $\mathbb{E}[T] < \infty$.

(ii) $(S_n)_{n\geq 0}$ is adapted to the filtration $(\mathcal{F}_n)_{n\geq 0}$ generated by $(X_n)_{n\geq 0}$ and then so are $(M_n)_{n\geq 0}$ and $(N_n)_{n\geq 0}$. They are integrable: clear for $(N_n)_{n\geq 0}$ and comes from $|S_n| \leq a + n$ for every $n \geq 0$ for $(M_n)_{n\geq 0}$. Then for every $n \geq 0$,

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{X_{n+1}}\right] = M_n \left[p\frac{1-p}{p} + (1-p)\frac{p}{1-p}\right] = M_n,$$

and

$$\mathbb{E}[N_{n+1} - N_n|\mathcal{F}_n] = \mathbb{E}[X_{n+1}] - (2p+1) = [p - (1-p)] - (2p-1) = 0.$$

(iii) We apply optional stopping Theorem to obtain that $(M_{n\wedge T})_{n\geq 0}$ is a martingale. Since $T < \infty$ a.s., we have $\lim_{n\to\infty} M_{n\wedge T} = M_T$ a.s.. We also notice that this process is bounded:

$$|M_{n\wedge T}| \leq \max\left(\left(\frac{1-p}{p}\right)^b, 1\right), \quad n \in \mathbb{N}.$$

By dominated convergence,

$$\mathbb{E}[M_T] = \lim_{n\to\infty} \mathbb{E}[M_{n\wedge T}] = \mathbb{E}[M_0].$$

That is,

$$\left(\frac{1-p}{p}\right)^a = \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{S_0}\right] = \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{S_T}\right] = \mathbb{P}(S_T = 0) + \left(\frac{1-p}{p}\right)^b \mathbb{P}(S_T = b).$$

Since $\mathbb{P}(S_T = b) = 1 - \mathbb{P}(S_T = 0)$, we get

$$\mathbb{P}(S_T = b) = \frac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^b - 1}, \quad \text{and} \quad \mathbb{P}(S_T = 0) = \frac{\left(\frac{1-p}{p}\right)^b - \left(\frac{1-p}{p}\right)^a}{\left(\frac{1-p}{p}\right)^b - 1}.$$

(iv) Recall that the process $(S_{n\wedge T})_{n\geq 0}$ is bounded by b. For every $n \geq 0$, we then get

$$|S_{n\wedge T} - (2p-1)n \wedge T| \leq b + (2p-1)T \in L^1.$$

Then from dominated convergence theorem

$$a = \mathbb{E}[N_0] = \mathbb{E}[N_{n\wedge T}] = \mathbb{E}[S_{n\wedge T} - (2p-1)n \wedge T] \to \mathbb{E}[S_T - (2p-1)T].$$

Finally

$$\mathbb{E}[T] = \frac{\mathbb{E}[S_T] - a}{2p-1} = \frac{b}{2p-1} \frac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^b - 1} - \frac{a}{2p-1}.$$

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