Wahrscheinlichkeitheorie 2 | **Universität Mannheim**
FSS 2020 | Prof. M. Slowik, Dr. Q. Shi

Sheet 4

For the exercise class 09/03/2020. Hand in your solutions before 17:00 Thursday 05/03/2020.

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$ for all the exercises. All the random variables are assumed to be real-valued.

Exercise 1. Let $(\Omega, \mathcal{F}, (\mathcal{F}_n, n \in \mathbb{N}_0), \mathbb{P})$ be a filtered probability space. Let $(T_k, k \in \mathbb{N})$ be a sequence of $(\mathcal{F}_n, n \in \mathbb{N}_0)$-stopping times. Show that $\inf_{k\in\mathbb{N}} T_k$, $\sup_{k\in\mathbb{N}} T_k$, $\liminf_{k\in\mathbb{N}} T_k$ and $\limsup_{k\in\mathbb{N}} T_k$ are all $(\mathcal{F}_n, n \in \mathbb{N}_0)$-stopping times.

Exercise 2. Let T be a $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$-stopping time. Suppose that there exists $N \in \mathbb{N}$ and $\varepsilon \in (0,1)$, such that for every $n \in \mathbb{N}$,

$$\mathbb{P}(T \leq n + N|\mathcal{F}_n) > \varepsilon, \quad \text{almost surely.}$$

(i) Prove that for each $k \in \mathbb{N}$, we have $\mathbb{P}(T > kN) \leq (1-\varepsilon)^k$.

Hint: $\mathbb{P}(T > kN) = \mathbb{P}(T > kN, T > (k-1)N)$.

(ii) Deduce that $\mathbb{E}[T] < \infty$.

Exercise 3. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. uniform random variables on $(0,1)$. Define for every $n \in \mathbb{N}$, $S_n := X_1 + \ldots + X_n$. Let $T := \inf\{n \geq 1 : S_n > 1\}$. Define S_T by:

$$S_T(\omega) = \begin{cases} S_n(\omega) & \text{if } T(\omega) = n \in \mathbb{N} \\ 0 & \text{if } T(\omega) = \infty. \end{cases} \quad \text{for every } \omega \in \Omega.$$

(i) Show that $\mathbb{P}(S_n \leq 1) = \frac{1}{n!}$.

(ii) Calculate $\mathbb{P}(T > n)$ and $\mathbb{E}[T]$.

(iii) Calculate $\mathbb{E}[S_T]$.

Exercise 4. The two questions are independent.

(i) Let $(X_n)_{n\in\mathbb{N}}$ be a $(\mathcal{F}_n)_{n\in\mathbb{N}}$-adapted stochastic process with $\mathbb{E}[X_n^2] < \infty$ for each $n \in \mathbb{N}$. Define for every $n \geq 1$, $S_n := X_1 + \ldots + X_n$. Suppose that $(S_n)_{n\in\mathbb{N}}$ is a $(\mathcal{F}_n)_{n\in\mathbb{N}}$-martingale. Show that if $i \neq j$, then $\mathbb{E}[X_i X_j] = 0$.

(ii) Let $(X_n)_{n\in\mathbb{N}_0}$ be a $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$-adapted stochastic process with $X_0 = 0$ and $\mathbb{E}[|X_n|] < \infty$ for each $n \in \mathbb{N}$. Suppose that there exists $a \in (0,1)$ such that $\mathbb{E}[X_{n+1}|X_1, \ldots, X_n] = aX_n + (1-a)X_{n-1}$ for every $n \geq 1$. Find a real number $b \in \mathbb{R}$ such that $S_n := bX_n + X_{n-1}$, $n \geq 1$, defines a $(\mathcal{F}_n)_{n\in\mathbb{N}}$-martingale.