Wahrscheinlichkeitheorie 2 **Universität Mannheim**
FSS 2020 Prof. M. Slowik, Dr. Q. Shi

Sheet 3

Hand in your solutions before 17:00 Friday 28/02/2020.

Exercise 1 (Monotone Class theorem for functions)**.** Let $(\Omega, \sigma(\mathcal{A}))$ be a measurable space, where $\mathcal{A}$ is a π-system (i.e. stable under intersection). Suppose $\mathcal{H}$ is a space of real-valued (bounded real valued, respectively) functions defined on Ω, that satisfies the following properties:

(i) $\mathcal{H}$ is a vector space: if $f, g \in \mathcal{H}$, then $cf + g \in \mathcal{H}$ for any $c \in \mathbb{R}$;

(ii) the constant function 1 belongs to $\mathcal{H}$;

(iii) if $(f_n, n \in \mathbb{N})$ is an increasing sequence of non-negative functions in $\mathcal{H}$ which converges pointwise to a non-negative (bounded non-negative, respectively) function f, then $f \in \mathcal{H}$.

(iv) every indicator function $\mathbf{1}_A$ with $A \in \mathcal{A}$ belongs to $\mathcal{H}$.

Prove that, $\mathcal{H}$ contains all real valued (bounded real valued, respectively) $(\sigma(\mathcal{A}), \mathcal{B}(\mathbb{R}))$-measurable functions.

Hint: Consider $\mathcal{S} := \{A \subset \Omega \colon \mathbf{1}_A \in \mathcal{H}\}$ and use $\pi - \lambda$ theorem (e.g. Theorem 1.19 in Klenke) to show that $\sigma(\mathcal{A}) \subset \mathcal{S}$.

Exercise 2. Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a filtration and $\mathcal{F}_\infty := \sigma(\mathcal{F}_n, n \in \mathbb{N}) = \sigma(\cup_{n\in\mathbb{N}}\mathcal{F}_n)$. Fix $p \in [1, \infty)$. Prove that, for every $\mathcal{F}_\infty$-measurable bounded random variable X, there is

$$\mathbb{E}[X \mid \mathcal{F}_n] \to \mathbb{E}[X \mid \mathcal{F}_\infty] \quad \text{in } \mathcal{L}^p.$$

Is the same statement true for every $\mathcal{F}_\infty$-measurable random variable X in $\mathcal{L}^p$?

Hint: note that $\cup_{n\in\mathbb{N}}\mathcal{F}_n$ is a π-system ($\cap$-stable). Use Exercise 1.