Wahrscheinlichkeitheorie 2 | **Universität Mannheim**
FSS 2020 | Prof. M. Slowik, Dr. Q. Shi

Sheet 2

For the exercise class 02.03.2020.
Hand in your solutions to B26 Floor 3, before 17:00 Thursday 20.02.2020.

Exercise 1. The three questions are independent.

(i) Let X and Y be i.i.d. Bernoulli variables: $\mathrm{P}(X=1) = 1-\mathrm{P}(X=0) = p$ for some $p \in [0,1]$. We set $Z := \mathbf{1}_{\{X+Y=0\}}$. Compute $\mathrm{E}[X|Z]$ and $\mathrm{E}[Y|Z]$. Are these random variables independent ?

(ii) Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra and $A \in \mathcal{F}$. We consider $B := \{\mathrm{E}[\mathbf{1}_A|\mathcal{G}] = 0\} \in \mathcal{G}$. Show that $B \subset \Omega \setminus A$ almost surely.

(iii) Let X be a square-integrable random variable and $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra. We define the *conditional variance*

$$\mathrm{Var}(X|\mathcal{G}) := \mathrm{E}[(X - \mathrm{E}[X|\mathcal{G}])^2|\mathcal{G}].$$

Show the following identity:

$$\mathrm{Var}(X) = \mathrm{E}[\mathrm{Var}(X|\mathcal{G})] + \mathrm{Var}(\mathrm{E}[X|\mathcal{G}]).$$

Exercise 2. Let X and Y be two integrable random variables.

(i) Assume that $X = Y$ almost surely. Prove that, almost surely, $\mathrm{E}[X|Y] = Y$ and $\mathrm{E}[Y|X] = X$.

(ii) Conversely, suppose that X and Y are such that $\mathrm{E}[X|Y] = Y$ and $\mathrm{E}[Y|X] = X$ almost surely. Prove that $X = Y$ almost surely.
Hint: You can consider $\mathrm{E}[(X-Y)\mathbf{1}_{\{X>c, Y\le c\}}] + \mathrm{E}[(X-Y)\mathbf{1}_{\{X\le c, Y \le c\}}]$.
(Recall that for any $x, y \in \mathbb{R}$, $x > y$ is equivalent to $\exists c \in \mathbb{Q}$ such that $x > c \geq y$.)

Exercise 3.

(i) Let X, Y, Z be three random variables such that (X, Z) and (Y, Z) are identically distributed. We denote by μ the distribution of X.

(a) Prove that for any measurable function f non-negative or such that $f(X)$ is integrable,

$$\mathrm{E}[f(X)|Z] = \mathrm{E}[f(Y)|Z] \quad \text{almost surely.}$$

(b) Let g be a measurable function non-negative or such that $g(Z)$ is integrable. Set

$$h_1(X) := \mathrm{E}[g(Z)|X] \quad \text{and} \quad h_2(Y) := \mathrm{E}[g(Z)|Y].$$

Show that $h_1 = h_2$ μ-almost surely.

(ii) Let $T_1, T_2, \ldots, T_n$ be i.i.d. integrable random variables and set $S_n = T_1 + \ldots + T_n$.

(a) Prove that for any $j \in \{1, \dots, n\}$, $\mathrm{E}[T_j|S_n] = n^{-1}S_n$ almost surely.

(b) Compute $\mathrm{E}[S_n|T_j]$ for any $j \in \{1, \dots, n\}$.

Exercise 4. Calculate the following conditional expectations and determine the conditional distributions.

(i) $\mathrm{E}\left[(Y-X)^+|X\right]$, where X, Y are independent uniform random variables on $[0, 1]$.

(ii) $\mathrm{E}\left[X|X+Y\right]$, where X, Y are independent Poisson random variables with parameters λ and μ respectively.

(iii) $\mathrm{E}\left[X|Y\right]$, where (X, Y) has a joint distribution with density:

$$4y(x-y)e^{-(x+y)}\mathbf{1}_{\{0<y<x\}}.$$

(iv) X, Y, Z are i.i.d. standard Gaussian random variables. $U = 2X - Y - Z$, $V = 3X + Y - 4Z$, calculate $\mathrm{E}\left[V|U\right]$.