Wahrscheinlichkeitheorie 2 | **Universität Mannheim**
FSS 2020 | Prof. M. Slowik, Dr. Q. Shi

Sheet 1

For the exercise class 17.02.2020.

For any set E, we denote by $\mathcal{P}(E)$ the powerset of E, i.e. the set of all subsets of E.

Exercise 1. Consider two measurable spaces $(\Omega, \mathcal{F})$ and $(E, \mathcal{E})$. Suppose that $\mathcal{E} = \sigma(\mathcal{A})$ with some $\mathcal{A} \subset \mathcal{P}(E)$. Show that a mapping $f\colon (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ is measurable, if

$$f^{-1}(A) \in \mathcal{F} \quad \text{for every } A \in \mathcal{A}.$$

Exercise 2. Consider two measurable spaces $(E, \mathcal{E})$ and $(F, \mathcal{F})$ and a function $f\colon E \to F$.

(i) Show that $\{B \subset F\colon f^{-1}B \in \mathcal{E}\}$ is a sigma-algebra.

(ii) Show that $\sigma(f) := \{f^{-1}B \subset E\colon B \in \mathcal{F}\}$ is a sigma-algebra.

(iii) Let $\mathcal{A} \subset \mathcal{P}(F)$. Then $f^{-1}(\sigma(\mathcal{A})) = \sigma(f^{-1}(\mathcal{A}))$.

Exercise 3. Let $E \subset \Omega$. Show that $\mathcal{F}_E := \{A \cap E\colon A \in \mathcal{F}\}$ is a sigma-algebra. If $\mathcal{F} = \sigma(\mathcal{E})$, that is $\mathcal{F}$ is generated by for $\mathcal{E}$, where $\mathcal{E}$ is a a collection of subsets of Ω. Then prove the identity $\mathcal{F}_E = \sigma(\{A \cap E\colon A \in \mathcal{E}\})$.

Exercise 4 (Factorization lemma)**.** Let $Y\colon (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ be measurable. Show that, for every random variable $X\colon (\Omega, \sigma(Y)) \to (\bar{\mathbb{R}} := [-\infty, +\infty], \mathcal{B}(\bar{\mathbb{R}}))$, there exists a measurable function $g\colon (E, \mathcal{E}) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$, such that $X = g(Y)$.

Solution. Reference of this exercise: Corollary 1.97 in Klenke.
We start with the case that X is a $\sigma(Y)$ simple function: that is $X = \sum_{i=1}^{k} \lambda_k \mathbf{1}_{A_k}$, where $\lambda_k \geq 0$ and $A_k \in \sigma(Y)$. By the definition of the sigma-algebra $\sigma(Y)$, the fact that $A_k \in \sigma(Y)$ implies that there exists $B_k \in \mathcal{E}$ such that $Y^{-1}(B_k) = A_k$. Define $g := \sum_{i=1}^{k} \lambda_k \mathbf{1}_{\{B_k\}}$, which is $(E, \mathcal{E}) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$-measurable. Then we have the identity, for every $\omega \in \Omega$:

$$X(\omega) = \sum_{i=1}^{k} \lambda_k \mathbf{1}_{\{A_k\}}(\omega) = \sum_{i=1}^{k} \lambda_k \mathbf{1}_{\{Y^{-1}(B_k)\}}(\omega) = \sum_{i=1}^{k} \lambda_k \mathbf{1}_{\{B_k\}}(Y(\omega)) = g(Y(\omega)).$$

This is the say $X = g(Y)$.
Now consider a non-negative $\sigma(Y)$-measurable function X. Then there exists a sequence of simple function $(X_n, n \in \mathbb{N})$, such that $X_n \leq X_{n+1}$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} X_n = X$. Applying the statement in the previous step, we have, for each $n \in \mathbb{N}$, a $(E, \mathcal{E}) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$-measurable function g_n, such that $X_n = g_n(Y)$. We define a function $g : (E, \mathcal{E}) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$ by

$$g(y) = \begin{cases} \lim_{n\to\infty} g_n(y), & \text{if exists or is } +\infty \\ 0, & \text{otherwises.} \end{cases}$$

Then g is a $(E,\mathcal{E}) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$-measurable function. Moreover, we have

$$X(\omega) = \lim_{n\to\infty} X_n(\omega) = \lim_{n\to\infty} g_n(Y(\omega)) = g(Y(\omega)), \quad \omega \in \Omega.$$

So we identify $X = g(Y)$.
Finally, we conclude for every $\sigma(Y)$-measurable function X by using the decomposition $X = X^+ - X^-$. □

Exercise 5. Recall that a gamma distribution with parameter $c > 0$ and $\theta > 0$ has density:

$$\frac{\theta^c}{\Gamma(c)} x^{c-1} \mathrm{e}^{-\theta x} \mathbf{1}_{\{x>0\}}.$$

(i) Check that the sum Z of two independent exponential random variables X, Y with parameter $\theta > 0$ has a gamma distribution with parameter $(2, \theta)$. Moreover, determine the conditional expectation of X given Z and prove that for every non-negative measurable function h, almost surely,

$$\mathbb{E}\left[h(X)|Z\right] = \frac{1}{Z}\int_0^Z h(u)\mathrm{d}u.$$

(ii) Conversely, let Z be a random variable with gamma distribution with parameter $(2, \theta)$, and suppose X is a random variable whose conditional distribution given Z is uniform on $[0, Z]$, in other words, for any h a non-negative measurable function,

$$\mathbb{E}\left[h(X)|Z\right] = \frac{1}{Z}\int_0^Z h(u)\mathrm{d}u, \quad \text{a.s.}$$

Prove that X and $Z - X$ are independent with distribution $\exp(\theta)$.

Solution.

(i) For $\forall g$ non-negative measurable function:

$$\begin{aligned}
& \mathbb{E}\left[h(X)g(Z)\right] \\
= & \mathbb{E}\left[h(X)g(X+Y)\right] \\
= & \int_{\mathbb{R}^2} h(x)g(x+y)\theta e^{-\theta x}\mathbf{1}_{\{x\geq 0\}}\theta e^{-\theta y}\mathbf{1}_{\{y\geq 0\}} dxdy \\
= & \int_{\mathbb{R}^2} h(x)g(z)\theta^2 e^{-\theta z}\mathbf{1}_{\{x\geq 0\}}\mathbf{1}_{\{z\geq x\}} dxdz \\
= & \int_{\mathbb{R}} g(z)\theta^2 e^{-\theta z} z\mathbf{1}_{\{z\geq 0\}} dz \left(\int_0^z \frac{1}{z}h(x)dx\right) \\
= & \mathbb{E}\left[g(Z)\left(\frac{1}{Z}\int_0^Z h(u)du\right)\right]
\end{aligned}$$

(ii) We first prove that $\forall h$ bounded $\mathcal{B}(\mathbb{R}^2)$-measurable function:

$$\mathbb{E}\left[h(X,Z)|Z\right] = \int_{\mathbb{R}} h(u,Z)\frac{\mathbf{1}_{\{0\leq u\leq Z\}}}{Z} du.$$

Let $\mathcal{H}$ be the class of bounded $\mathcal{B}(\mathbb{R}^2)$-measurable function that makes this identity hold. $\mathcal{H}$ is clearly a vector space that contains constant functions. We can easily check that $\mathcal{H}$ contains any indicator function on a rectangle in $\mathbb{R}^2$; i.e. $h(x,y) = \mathbf{1}_{\{x\in(a,b)\}}\mathbf{1}_{\{y\in(a',b')\}}$. Note that rectangles $\mathbb{R}^2$ form a π-system that generates the Borel sigma-algebra (see Theorem 1.23 Klenke). Using monotone convergence theorem (for integrals and for conditional expectations), we can prove that if $h_n \uparrow h$ with $h_n \in \mathcal{H}$ and h bounded, then $h \in \mathcal{H}$. So we have justifies the conditions in monotone class theorems for functions. Therefore, we conclude that the identity holds for all $\mathcal{B}(\mathbb{R}^2)$-measurable function. Thus we know $\forall f, g$ bounded measurable functions:

$$\begin{aligned}
\mathbb{E}\left[f(X)g(Z-X)\right] &= \mathbb{E}\left[\mathbb{E}\left[f(X)g(Z-X)|Z\right]\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}} f(u)g(Z-u)\frac{\mathbf{1}_{\{0\le u\le Z\}}}{Z}du\right] \\
&= \int_{\mathbb{R}^2} f(u)g(z-u)\frac{\mathbf{1}_{\{0\le u\le z\}}}{z}\theta^2 z e^{-\theta z}\mathbf{1}_{\{z\ge 0\}}dudz \\
&= \int_{\mathbb{R}^2} f(u)g(y)\theta^2 e^{-\theta(u+y)}\mathbf{1}_{\{y\ge 0\}}\mathbf{1}_{\{u\ge 0\}}dudy
\end{aligned}$$

which shows that $(X, Z-X)$ are independent $exp(\theta)$ random variables.

□

Exercise 6. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra.

(i) Prove that if $\mathbb{E}\left[X^2\right] < \infty$ and $\mathbb{E}\left[X|\mathcal{G}\right]$ has the same distribution as X, then $\mathbb{E}\left[X|\mathcal{G}\right] = X$ a.s.
Hint: You can prove that $\mathbb{E}\left[\mathbb{E}\left[X|\mathcal{G}\right]^2\right] = \mathbb{E}\left[X^2\right]$.

(ii) Prove (i) under assumption $\mathbb{E}\left[|X|\right] < \infty$ (instead of $\mathbb{E}\left[X^2\right] < \infty$).
Hint: We may consider $\mathbb{E}\left[|Y| - Y; \mathbb{E}\left[Y|\mathcal{G}\right] > 0\right]$ in order to prove $\operatorname{sgn}(Y) = \operatorname{sgn}(\mathbb{E}\left[Y|\mathcal{G}\right])$ a.s.; then take $Y = X - c$ for all rational c to get the desire conclusion.

Solution.

(i) if X and $\mathbb{E}\left[X|\mathcal{G}\right]$ have same law, then

$$\mathbb{E}\left[X^2\right] = \mathbb{E}\left[\mathbb{E}\left[X|\mathcal{G}\right]^2\right].$$

We also have

$$\mathbb{E}\left[X\mathbb{E}\left[X|\mathcal{G}\right]\right] = \mathbb{E}\left[\mathbb{E}\left[X\mathbb{E}\left[X|\mathcal{G}\right]|\mathcal{G}\right]\right] = \mathbb{E}\left[\mathbb{E}\left[X|\mathcal{G}\right]^2\right].$$

Then

$$\mathbb{E}\left[(X - \mathbb{E}\left[X|\mathcal{G}\right])^2\right] = \mathbb{E}\left[X^2\right] - 2\mathbb{E}\left[X\mathbb{E}\left[X|\mathcal{G}\right]\right] + \mathbb{E}\left[\mathbb{E}\left[X|\mathcal{G}\right]^2\right] = 0.$$

It follows that $X = \mathbb{E}\left[X|\mathcal{G}\right]$ a.s.

(ii) Fix $c \in \mathbb{Q}$. Let $Y := X - c$, then Y and $\mathbb{E}\left[Y|\mathcal{G}\right]$ have the same law. So we have $\mathbb{E}\left[|Y|\right] = \mathbb{E}\left[|\mathbb{E}\left[Y|\mathcal{G}\right]|\right]$ and it follows that

$$\mathbb{E}\Big[\mathbb{E}\left[|Y||\mathcal{G}\right] - \left|\mathbb{E}\left[Y|\mathcal{G}\right]\right|\Big] = \mathbb{E}\left[|Y|\right] - \mathbb{E}\left[\left|\mathbb{E}\left[Y|\mathcal{G}\right]\right|\right] = 0.$$

By Jensen's inequality, we also have:

$$\mathbb{E}\left[|Y||\mathcal{G}\right] \geq |\mathbb{E}\left[Y|\mathcal{G}\right]|, \quad \mathbb{P}\text{-a.s..}$$

So we deduce that $\mathbb{E}\left[|Y||\mathcal{G}\right] = |\mathbb{E}\left[Y|\mathcal{G}\right]|$ $\mathbb{P}$-a.s..

Then we have:

$$\begin{aligned}
\mathbb{E}\left[|Y| - Y; \mathbb{E}\left[Y|\mathcal{G}\right] \geq 0\right] &= \mathbb{E}\left[|Y|\mathbf{1}_{\{\mathbb{E}[Y|\mathcal{G}]\geq 0\}} - Y\mathbf{1}_{\{\mathbb{E}[Y|\mathcal{G}]>\geq 0\}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[|Y|\mathbf{1}_{\{\mathbb{E}[Y|\mathcal{G}]\geq 0\}}|\mathcal{G}\right]\right] - \mathbb{E}\left[\mathbb{E}\left[Y\mathbf{1}_{\{\mathbb{E}[Y|\mathcal{G}]\geq 0\}}|\mathcal{G}\right]\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{\mathbb{E}[Y|\mathcal{G}]\geq 0\}}\mathbb{E}\left[|Y||\mathcal{G}\right]\right] - \mathbb{E}\left[\mathbf{1}_{\{\mathbb{E}[Y|\mathcal{G}]\geq 0\}}\mathbb{E}\left[Y|\mathcal{G}\right]\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{\mathbb{E}[Y|\mathcal{G}]\geq 0\}}\mathbb{E}\left[|Y||\mathcal{G}\right]\right] - \mathbb{E}\left[\mathbf{1}_{\{\mathbb{E}[Y|\mathcal{G}]\geq 0\}}|\mathbb{E}\left[Y|\mathcal{G}\right]|\right] \\
&= 0.
\end{aligned}$$

Define a function: $sgn(y) = 1$ when $y \geq 0$ and $sgn(y) = -1$ when $y < 0$. Then the calculation above implies that (up to a $\mathbb{P}$-negligible set difference)

$$\{\mathbb{E}\left[Y|\mathcal{G}\right] \geq 0\} \subset \{|Y| - Y = 0\} = \{Y \geq 0\}. \tag{1}$$

Similarly, we also have $\mathbb{E}\left[|Y| + Y; \mathbb{E}\left[Y|\mathcal{G}\right] \leq 0\right] = 0$, which implies

$$\{\mathbb{E}\left[Y|\mathcal{G}\right] \leq 0\} \subset \{|Y| + Y = 0\} = \{Y \leq 0\}. \tag{2}$$

Combining (1) and (2), we have:

$$sgn(Y) = sgn(\mathbb{E}\left[Y|\mathcal{G}\right])\ a.s.$$

$\forall c \in \mathbb{Q}$, replace Y by $X - c$, we have $sgn(X - c) = sgn(\mathbb{E}\left[X|\mathcal{G}\right] - c)$ a.s.. As a consequence ($\mathbb{Q}$ is countable), $\mathcal{P}$-a.s. $sgn(X - c) = sgn(\mathbb{E}\left[X|\mathcal{G}\right] - c)$ holds for all $c \in \mathbb{Q}$. We conclude that $X = \mathbb{E}\left[X|\mathcal{G}\right]$ a.s. .

$\square$