Wahrscheinlichkeitstheorie 2 | **Universität Mannheim**
FSS 2020 | Dr. Q. Shi

Sheet 13

Hand in your solutions before 17:00 Thursday 28/May/2020.

We consider a Markov chain X_n that takes values in a countable set E. Let Q denote its transition probability. Let $Q^n(x, y) := \mathbb{P}(X_n = y | X_0 = x) = \mathbb{P}_x(X_n = y)$, for all $n \geq 1$.
For $x \in E$, the hitting time $H_x = \inf\{n \geq 1 : X_n = x\}$.

Exercise 1. We consider a Markov chain on state space $\{1, 2, \dots, 7\}$ with transition matrix

$$\big(Q(i,j), i,j \in \{1,\dots,7\}\big) = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 & 0 & 0 \\ 2/7 & 1/7 & 0 & 1/7 & 0 & 1/7 & 2/7 \\ 0 & 0 & 0 & 0 & 1/5 & 0 & 4/5 \\ 0 & 0 & 2/7 & 1/7 & 4/7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/5 & 0 & 3/5 \end{pmatrix}$$

Identify the irreducible classes. Classify the states (recurrent or transient; if recurrent, positive recurrent or null recurrent). Give all their stationary measures $(\mu_i, i = 1, \cdots, 7)$.

Exercise 2. We consider a Markov chain on $\mathbb{N}_0$, with $Q(i, i+1) = p_i \in (0, 1)$, $Q(i, 0) = 1 - p_i$, for all $i \geq 0$.

(i) Discuss the classification of states for this Markov chain.
Hint: Calculate $\mathbb{P}_0(H_0 > i)$, then $\mathbb{P}_0(H_0 = \infty)$. Find a necessary and sufficient condition for recurrence.

(ii) Prove that in the transient case, the only invariant measure satisfying $\mu = \mu Q$ is $\mu_i = 0, \forall i \geq 0$.

(iii) In the recurrent case, find the invariant measures and calculate $\mathbb{E}_0[H_0]$. Find a necessary and sufficient condition for positive recurrence.

Exercise 3. Let $(Y_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with values in $\mathbb{N} = \{1, 2, 3, \dots\}$ and such that

$$\mathbb{E}[Y_1] < \infty \quad \text{and} \quad \gcd\{n \geq 1 : \mathbb{P}(Y_1 = n) > 0\} = 1,$$

where gcd means the greatest common divisor. We define a process X by $X_0 := 0$ and for every $n \geq 1$,

$$X_n := \inf\{m \geq n : \exists k \geq 1, Y_1 + \ldots + Y_k = m\} - n.$$

In other word, X_n is the distance from n to the first element of $\{Y_1 + \ldots + Y_k; k \geq 1\}$ located to the right of n (or on n). Show that X is a Markov chain and that it is irreducible, positive recurrent and aperiodic. Deduce that

$$\lim_{n\to\infty} \mathbb{P}(\exists k \geq 1, Y_1 + \ldots + Y_k = n) = \frac{1}{\mathbb{E}[Y_1]}.$$

Exercise 4. Suppose that the Markov chain is irreducible and there exists an invariant distribution π. Let μ be a probability on E and $\mathcal{T}$ a stopping time such that

$$\mathcal{T} > 0, \quad \mathbb{E}[\mathcal{T}] < \infty, \quad \mathbb{P}_\mu(X_{\mathcal{T}} \in \cdot) = \mu(\cdot).$$

Define a measure ρ on E by

$$\rho(y) = \mathbb{E}_\mu\Big[\sum_{k=0}^{\mathcal{T}-1} \mathbf{1}_{\{X_k = y\}}\Big], \qquad y \in E.$$

Show that ρ is an invariant measure. Deduce that for any $y \in E$, we have

$$\rho(y) = \pi(y)\mathbb{E}_\mu[\mathcal{T}].$$

Remark: with $\mu = \delta_x$ and $\mathcal{T} = H_x$, we rediscover the invariant measure constructed in the lecture notes.

Exercise 5. For all $x \in E$ and $n \in \mathbb{N}$, we define

$$N(x, n) \coloneqq \sum_{k=0}^{n-1} \mathbf{1}_{\{X_k = x\}}.$$

Suppose that the Markov chain is irreducible and there exists an invariant distribution π. Show that for all $x, y, z \in E$, we have

(i) $\mathbb{E}_x[H_x] = \dfrac{1}{\pi(x)}$ and $\mathbb{E}_x[N(y, H_x)] = \dfrac{\pi(y)}{\pi(x)}$.

(ii) If $y \neq x$, then $\mathbb{E}_y[N(y, H_x)] = \pi(y)(\mathbb{E}_y[H_x] + \mathbb{E}_x[H_y])$.

(iii) Deduce that if $y \neq x$, then $\mathbb{P}_y(H_x < H_y) = \dfrac{1}{\pi(y)(\mathbb{E}_x[H_y] + \mathbb{E}_y[H_x])}$.

(iv) If $x \neq z$ and $y \neq z$, then $\mathbb{E}_x[N(y, H_z)] = \pi(y)(\mathbb{E}_x[H_z] + \mathbb{E}_z[H_y] - \mathbb{E}_x[H_y])$.

(v) Deduce that if $x \neq z$ and $y \neq z$, then $\mathbb{P}_x(H_y < H_z) = \dfrac{\mathbb{E}_x[H_z] + \mathbb{E}_z[H_y] - \mathbb{E}_x[H_y]}{\mathbb{E}_y[H_z] + \mathbb{E}_z[H_y]}$.

Hint: You may use Exercise 4 with appropriate μ and $\mathcal{T}$.