Sheet 12

Hand in your solutions before 17:00 Thursday 21/May/2020.

We consider a Markov chain X_n that takes values in a countable set E. Let Q denote its transition probability. Let $Q^n(x,y) := \mathbb{P}\left(X_n = y | X_0 = x\right) = \mathbb{P}_x\left(X_n = y\right)$, for all $n \geq 0$. Let θ_k be the shift operator on the product space $E^{\mathbb{N}_0}$: $\theta_k((\omega_n)_{n\in\mathbb{N}_0}) = (\omega_{k+n})_{n\in\mathbb{N}_0}$.

Exercise 1. Various (and quick) questions on state classification. We denote by $N_x = \sum_{n=0}^{\infty} \mathbf{1}_{\{X_n = x\}}$. Justify your answers by giving a proof or a counterexample.

(i) Give an example such that the set of points visited by the chain issued from a point x is not almost surely constant.

(ii) Give an example such that the set of points visited by the chain issued from a point x is almost surely constant, the order of the first three points visited is not deterministic, and such that x is not recurrent.

(iii) Let $x, y \in E$. Do we have: y is recurrent and there exists $n \geq 0$ such that $Q^n(x,y) > 0$ implies $N_y = \infty$, $\mathbb{P}_x$ almost surely?

(iv) Give an example where there exists $n \geq 0$ such that $Q^n(x,y) > 0$ but for all $p \geq 0$, $Q^p(y,x) = 0$.

(v) Let $x, y \in E$. Show that $\mathbb{E}_x[N_y] = \infty \implies y$ is recurrent. Is it equivalent?

(vi) Is the following situation possible: $0 < \mathbb{E}_x[N_y] < \infty$ and y is recurrent?

(vii) If $\mathbb{E}_x[N_y] = \infty$, which values can take $\mathbb{E}_y[N_x]$?

(viii) Suppose that for every $x \in E$, the set $V_x := \{y \in E : \exists n \text{ s.t. } Q^n(x,y) > 0\}$ is finite. Show that there exists recurrent states.

(ix) Suppose that there exists $x_0 \in E$ such that for every $x \in E$, there exists $n_x \geq 0$, $Q^{n_x}(x_0, x) > 0$ and $\mathbb{P}_x(T_{x_0} < \infty) = 1$, where $T_{x_0} = \inf\{n \geq 0 : X_n = x_0\}$. Is the chain recurrent ?

Exercise 2. Show that X is irreducible if and only if there exist no strict, non-empty, subset F of E such that

$$\text{for all } x \in F, y \in F^c, \qquad Q(x,y) = 0.$$

Exercise 3 (Gambler's Ruin)**.** Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{P}(X_1 = 1) = 1 - \mathbb{P}(X_1 = -1) = p$ for some $p \in (0,1)$. Given X_0, then $S_n := \sum_{i=0}^{n} X_i$ is a Markov chain.

(i) Define $T_0 := \inf\{n \geq 0 : S_n = 0\}$. for all $i \geq 0$, let $h(i) = \mathbb{P}_i\left(T_0 < \infty\right)$. Calculate $h(i)$, for all $i \geq 0$. **Hint**: you can use Ex2 in the additional sheet.

(ii) Given an arbitrary number $s \in (0,1)$, we notice that $s^{T_0} = s^{T_0}\mathbf{1}_{\{T_0<\infty\}}$. Prove that:

$$\mathbb{E}_1\left[s^{T_0}\right] = ps\mathbb{E}_2\left[s^{T_0}\right] + qs.$$

(iii) Given an arbitrary number $s \in (0,1)$, explain why $\mathbb{E}_2\left[s^{T_1}\right] = \mathbb{E}_1\left[s^{T_0}\right]$, and prove that:

$$\mathbb{E}_2\left[s^{T_0}\right] = \mathbb{E}_1\left[s^{T_0}\right]^2.$$

Hint: Use strong Markov property for T_1.

(iv) We define a function $\phi : (0,1) \to (0,1)$ by $\phi(s) = \mathbb{E}_1\left[s^{T_0}\right]$. Prove that $\phi(s)$ satisfies the following equation:

$$\phi(s) = ps\phi^2(s) + qs.$$

(v) Prove that $\lim_{s\to1^-}\phi(s) = \mathbb{P}_1(T_0 < \infty)$, and $\lim_{s\to1^-}\phi'(s) = \mathbb{E}_1\left[T_0\mathbf{1}_{\{T_0<\infty\}}\right]$, then calculate their values.

Exercise 4. This exercise will help us identify recurrent states. We define "$x \to y$" if $\mathbb{P}_x(T_y < \infty) > 0$; "$x \sim y$" if and only if $x \to y$ and $y \to x$. Clearly $x \sim y$ is an equivalence relation. Thus E is a union of disjoint equivalence classes under relation $\sim$:

$$E = \bigcup_i E_i,$$

such that for all i, if $x, y \in E_i$, then $x \sim y$; if $i \neq j$, and $x \in E_i$, $y \in E_j$, then $x \not\sim y$.
We call each E_i an *irreducible class*. E_i is called a *closed* class if for all $x \in E_i$, if $x \to y$, then $y \in E_i$.

(i) Show that in an irreducible class, all the states are recurrent or all the states are transient. So we can say the class is recurrent or transient.

(ii) Show that any recurrent class is closed.

(iii) Show that a finite closed class is recurrent.