Sheet 11

Hand in your solutions before 17:00 Thursday 07/May/2020. Exercises 4 and 5 are optional, not for homework.

We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for all the exercises. The process $(B_t)_{t\geq 0}$ denotes a real-valued Brownian motion defined on this space. Let $\mathcal{F}_t = \sigma(B_s, s \leq t)$.

Exercise 1. For $a, b > 0$, let $X_t = B_t - bt$ and $\tau = \inf\{t > 0 \colon X_t = a\}$.

(i) For every $\gamma \in \mathbb{R}$, show that the process $(M_t = \exp(\gamma B_t - \gamma^2 t/2), t \geq 0)$ is a martingale.

(ii) Let $\lambda > 0$. Calculate $\mathbb{E}[\exp(-\lambda\tau)\mathbf{1}_{\{\tau<\infty\}}]$.
hint: use the martingale in (i) with γ being the positive solution of $\gamma^2 - 2b\gamma - 2\lambda = 0$.

(iii) Calculate $\mathbb{P}(\tau < \infty)$.

Exercise 2. Let $\mathcal{Z} := \{t \geq 0 : B_t = 0\}$ be the zero set of a Brownian motion. Show that, almost surely,

(i) $\mathcal{Z}$ is closed,

(ii) $\mathcal{Z}$ is unbounded,

(iii) $\mathcal{Z}$ has zero Lebesgue measure,

(iv) $\mathcal{Z}$ has no isolated point.

Hint 1: you may assume that the map

$$\begin{array}{rccc} \phi: & \Omega \times \mathbb{R}_+ & \longrightarrow & \{0,1\} \\ & (\omega, t) & \longmapsto & \mathbf{1}_{\{B_t \neq 0\}} \end{array}$$

is measurable when $\Omega \times \mathbb{R}_+$ is equipped with the σ-algebra $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$.
Hint 2: you may consider for every $q \in \mathbb{Q}_+$ the stopping time $d_q := \inf\{t \geq q : B_t = 0\}$ and show that, almost surely, for every $q \in \mathbb{Q}_+$, $d_q < \infty$ and $d_q \in \mathcal{Z}$ is not isolated. Then approach every element of $\mathcal{Z}$ by such d_q's.

Exercise 3. We define for every $t \geq 0$, $S_t := \sup_{s\in[0,t]} B_s$ the running maximum of a Brownian motion and two random times

$$T := \inf\{t \in [0,1] : B_t = S_1\} \quad \text{and} \quad L := \sup\{t \in [0,1] : B_t = 0\}.$$

We admit that $\inf\{t \geq 0 : B_t = a\} = a^2/B_1^2$ in distribution for every $a > 0$ and $S_t - B_t \overset{(d)}{=} S_t \overset{(d)}{=} |B_t|$ in distribution for every $t \in [0,1]$, which follows from the reflection principle.

(i) Show that the processes $(B_t)_{t\in[0,1]}$ and $(B_{1-t} - B_1)_{t\in[0,1]}$ are equally distributed.

(ii) Show that $T, L \in (0,1)$ almost surely.

(iii) Prove that T and L are not stopping times. (you may prove by contradiction: if they are stopping times, then strong Markov property holds)

(iv) Show that L has the following density with respect to the Lebesgue measure:

$$\frac{1}{\pi\sqrt{t(1-t)}}\mathbf{1}_{(0,1)}(t).$$

Hint: use simple Markov property to show that

$$\mathbb{P}(L \le t) = \int_{\mathbb{R}} \mathbb{P}(B_t \in dx)\mathbb{P}(\inf\{s > 0 : x + B_s = 0\} > 1 - t).$$

(v) Show that T and L have the same distribution.

Exercise 4 (Optional)**.** (i) Fix any $t \ge 0$. Prove that

$$\int_0^t \mathrm{e}^{B_s}\mathrm{d}s \overset{(\mathrm{d})}{=} t\int_0^1 \mathrm{e}^{\sqrt{t}B_s}\mathrm{d}s.$$

(ii) We denote by $S_1 \coloneqq \sup_{t\in[0,1]} B_t$. Show that

$$\lim_{t\to\infty}\left(\int_0^1 \mathrm{e}^{\sqrt{t}B_s}\mathrm{d}s\right)^{1/\sqrt{t}} = \mathrm{e}^{S_1} \quad \text{almost surely.}$$

(iii) Prove that

$$\lim_{t\to\infty}\left(\int_0^t \mathrm{e}^{B_s}\mathrm{d}s\right)^{1/\sqrt{t}} = \mathrm{e}^{S_1} \quad \text{in distribution.}$$

Exercise 5 (Optional)**.** On local maxima of Brownian motion.

(i) Let $0 \le p < q < r < s < \infty$. Show that

$$\mathbb{P}\left(\sup_{p\le t\le q} B_t = \sup_{r\le t\le s} B_t\right) = 0.$$

(ii) Deduce that almost surely, the local maxima of $t \mapsto B_t$ are all distinct.

(iii) Show that, almost surely, the set of times where B realizes the local maxima is at most countable and dense in $[0,\infty)$.
Hint: you can show that if B realizes a local maximum at time t, then t belongs to the image of the map

$$(a,b) \mapsto \inf\{t \ge a : B_t = \sup_{s\in[a,b]} B_s\}, \quad 0 \le a < b \text{ rational numbers.}$$