Wahrscheinlichkeitstheorie 2 **Universität Mannheim**
FSS 2020 Dr. Q. Shi

Sheet 10

Hand in your solutions before 17:00 Thursday 30/April/2020.

Exercise 1. Let $(X_t, t \geq 0)$ be a continuous stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. For $0 \leq a < b < \infty$, define a function $\int_a^b X_t dt$ from Ω to $\mathbb{R}$:

$$\omega \mapsto \int_a^b X_t(\omega) dt := \begin{cases} \int_a^b X_t(\omega)dt, & \text{if } (X_t(\omega), t \geq 0) \text{ is conitnuous;} \\ 0, & \text{otherwises.} \end{cases}$$

Show that $\int_a^b X_t dt$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$: i.e. this function is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable.

Exercise 2. A stochastic process $(X_t)_{0 \leq t \leq 1}$ is called a *Brownian bridge* if:

- it is a centred Gaussian process with covariance function given by

$$\mathrm{Cov}(X_s, X_t) = s(1-t), \quad s \leq t,$$

- and it is a.s. continuous.

(i) Let $(B_t, t \geq 0)$ be a standard Brownian motion. Show that the process

$$X_t = B_t - tB_1, \quad t \in [0,1]$$

is a Brownian bridge. Moreover, prove that B_1 is independent from $(X_t, t \in [0,1])$.

(ii) Let $(B_t, t \geq 0)$ be a standard Brownian motion. Show that the process

$$X_t = (1-t) B_{\frac{t}{1-t}}, \quad t \in [0,1].$$

is a Brownian bridge.

Exercise 3. Let $(B_t, t \geq 0)$ be a standard Brownian motion.

(i) For an arbitrary $t > 0$, calculate $\mathbb{E}\left[B_t^4\right]$ and $\mathbb{E}\left[|B_t|\right]$.

(ii) For an arbitrary $t > 0$, and $a \in \mathbb{R}$, calculate $\mathbb{E}\left[\exp(aB_t)\right]$.

(iii) Let $\mathcal{F}_1 := \sigma(B_s, s \in [0,1])$. Calculate $\mathbb{E}\left[B_5 | \mathcal{F}_1\right]$ and $\mathbb{E}\left[B_5^2 | \mathcal{F}_1\right]$.

(iv) Show that $\mathbb{E}\left[\left| \int_{[0,1]} B_s/s \, ds \right|\right] < \infty$.

(v) Let $\beta_t := B_t - \int_{[0,t]} \frac{B_s}{s} ds$. Show that $(\beta_t, t \geq 0)$ is a Brownian motion.

Exercise 4. A partition $\Pi = (t_0, \cdots, t_n)$ of an interval $[a, b]$ is any sequence of values $a = t_0 < t_1 < \cdots t_{n-1} < t_n = b$. We fix an arbitrary $[a, b]$, such that $0 \leq a < b$. For $n \in \mathbb{N}$, we pose a partition Π_n for $[a, b]$:

$$\Pi_n = (a, a+(b-a)2^{-n}, \cdots, a+k(b-a)2^{-n}, a+(k+1)(b-a)2^{-n}, \cdots, a+(2^n-1)(b-a)2^{-n}, b)$$

Let $(B_t, t \geq 0)$ be a standard Brownian motion.

(i) Define $\Delta_{k,n} = B_{a+k(b-a)2^{-n}} - B_{a+(k-1)(b-a)2^{-n}}$ for $k = 1, \cdots, 2^n$. Prove that the sequence $(\Delta_{k,n}, k = 1, \cdots, 2^n)$ is a sequence of i.i.d. random variables with Gaussian distribution $\mathcal{N}(0, (b-a)2^{-n})$.

(ii) Let X_n be a random variable defined by:

$$X_n = \sum_{k=1}^{2^n} \Delta_{k,n}^2 = \sum_{k=1}^{2^n} (B_{a+k(b-a)2^{-n}} - B_{a+(k-1)(b-a)2^{-n}})^2$$

Calculate $\mathbb{E}[X_n]$ and $\mathrm{Var}(X_n)$.

(iii) Prove that $X_n \xrightarrow[n\to\infty]{a.s.} b - a$.

(iv) Given a continuous function $f : [a, b] \to \mathbb{R}$, its total variation on $[a, b]$ is defined to be

$$\mathrm{TV}(f) := \sup_{\Pi} \sum_{1 \leq k \leq n} \left| f(t_k) - f(t_{k-1}) \right| \in [0, \infty],$$

where the supremum runs over the set of all partitions.
Show that almost surely, the path of a Brownian motion has infinite total variation on any interval $[a, b]$.
Hint: Use the continuity of a Brownian motion and (iii).