Wahrscheinlichkeitheorie 2 | **Universität Mannheim**
FSS 2020 | Prof. M. Slowik, Dr. Q. Shi

Sheet 1

For the exercise class 17.02.2020.

For any set E, we denote by $\mathcal{P}(E)$ the powerset of E, i.e. the set of all subsets of E.

Exercise 1. Consider two measurable spaces $(\Omega, \mathcal{F})$ and $(E, \mathcal{E})$. Suppose that $\mathcal{E} = \sigma(\mathcal{A})$ with some $\mathcal{A} \subset \mathcal{P}(E)$. Show that a mapping $f\colon (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ is measurable, if

$$f^{-1}(A) \in \mathcal{F} \quad \text{for every } A \in \mathcal{A}.$$

Exercise 2. Consider two measurable spaces $(E, \mathcal{E})$ and $(F, \mathcal{F})$ and a function $f\colon E \to F$.

(i) Show that $\{B \subset F\colon f^{-1}B \in \mathcal{E}\}$ is a sigma-algebra.

(ii) Show that $\sigma(f) := \{f^{-1}B \subset E\colon B \in \mathcal{F}\}$ is a sigma-algebra.

(iii) Let $\mathcal{A} \subset \mathcal{P}(F)$. Then $f^{-1}(\sigma(\mathcal{A})) = \sigma(f^{-1}(\mathcal{A}))$.

Exercise 3. Let $E \subset \Omega$. Show that $\mathcal{F}_E := \{A \cap E\colon A \in \mathcal{F}\}$ is a sigma-algebra. If $\mathcal{F} = \sigma(\mathcal{E})$, that is $\mathcal{F}$ is generated by for $\mathcal{E}$, where $\mathcal{E}$ is a a collection of subsets of Ω. Then prove the identity $\mathcal{F}_E = \sigma(\{A \cap E\colon A \in \mathcal{E}\})$.

Exercise 4. Let $Y\colon (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be measurable. Show that, for every random variable $X\colon (\Omega, \sigma(Y)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, there exists a measurable function $g\colon (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, such that $X = g(Y)$.

Exercise 5. Recall that a gamma distribution with parameter $c > 0$ and $\theta > 0$ has density:

$$\frac{\theta^c}{\Gamma(c)} x^{c-1} \mathrm{e}^{-\theta x} \mathbf{1}_{\{x>0\}}.$$

(i) Check that the sum Z of two independent exponential random variables X, Y with parameter $\theta > 0$ has a gamma distribution with parameter $(2, \theta)$. Moreover, determine the conditional expectation of X given Z and prove that for every non-negative measurable function h, almost surely,

$$\mathrm{E}\left[h(X) | Z\right] = \frac{1}{Z} \int_0^Z h(u) \mathrm{d}u.$$

(ii) Conversely, let Z be a random variable with gamma distribution with parameter $(2, \theta)$, and suppose X is a random variable whose conditional distribution given Z is uniform on $[0, Z]$, in other words, for any h a non-negative measurable function,

$$\mathrm{E}\left[h(X) | Z\right] = \frac{1}{Z} \int_0^Z h(u) \mathrm{d}u, \quad \text{a.s.}$$

Prove that X and $Z - X$ are independent with distribution $\exp(\theta)$.

Exercise 6. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra.

(i) Prove that if $\mathrm{E}\left[X^2\right] < \infty$ and $\mathrm{E}\left[X|\mathcal{G}\right]$ has the same distribution as X, then $\mathrm{E}\left[X|\mathcal{G}\right] = X$ a.s.
Hint: You can prove that $\mathrm{E}\left[\mathrm{E}\left[X|\mathcal{G}\right]^2\right] = \mathrm{E}\left[X^2\right]$.

(ii) Prove (i) under assumption $\mathrm{E}\left[|X|\right] < \infty$ (instead of $\mathrm{E}\left[X^2\right] < \infty$).
Hint: We may consider $\mathrm{E}\left[|Y| - Y; \mathrm{E}\left[Y|\mathcal{G}\right] > 0\right]$ in order to prove $\mathrm{sgn}(Y) = \mathrm{sgn}(\mathrm{E}\left[Y|\mathcal{G}\right])$ a.s.; then take $Y = X - c$ for all rational c to get the desire conclusion.