Prf. • We first check $X_T \in \mathcal{L}^1$:

$$E[|X_T|] = \sum_{n=0}^{+\infty} E[\mathbb{1}_{\{T=n\}} |X_n|] + E[\mathbb{1}_{\{T=\infty\}} |X_\infty|]$$

$$= \sum_{n=0}^{+\infty} E[\mathbb{1}_{\{T=n\}} |E[X_\infty | \mathcal{F}_n]|] + E[\mathbb{1}_{\{T=\infty\}} |X_\infty|].$$

Jensen

$$\leq \sum_{n=0}^{+\infty} E[\mathbb{1}_{\{T=n\}} E[|X_\infty| | \mathcal{F}_n]] + \text{———}$$

$$= \sum_{n=0}^{+\infty} E[|X_\infty| \cdot \mathbb{1}_{\{T=n\}}] + E[\mathbb{1}_{\{T=\infty\}} |X_\infty|] = E[|X_\infty|] < \infty.$$

• For $\forall A \in \mathcal{F}_T$: $A \cap \{T=n\} \in \mathcal{F}_n$

~~$E[\mathbb{1}_A X_T] =$~~ $E[\mathbb{1}_{A\cap\{T=n\}} X_T] \underset{\uparrow}{=} E[\mathbb{1}_{A\cap\{T=n\}} X_n] \underset{\uparrow}{=} E[\mathbb{1}_{A\cap\{T=n\}} X_\infty]$

T=n event $\qquad$ $E[X_\infty | \mathcal{F}_n] = X_n$.

$$\Rightarrow E[\mathbb{1}_A X_T] = \sum_{n=0}^{+\infty} E[\mathbb{1}_{A\cap\{T=n\}} X_T] + E[\mathbb{1}_{A\cap\{T=\infty\}} X_\infty] = E[\mathbb{1}_A X_\infty]$$

i.e. $E[X_\infty | \mathcal{F}_T] = X_T$. ($\because \mathcal{F}_T$.)

• $S \leq T$ stopping Times: $X_S = E[X_\infty | \mathcal{F}_S] = E[E[X_\infty | \mathcal{F}_T] | \mathcal{F}_S]$

$$= E[X_T | \mathcal{F}_S] \quad \#$$

Thm 29 (Optional Stopping Thm for ~~sub~~ supermartingales).

$(X_n)_{n\in\mathbb{N}}$ supermartingale. Suppose either (i) or (ii):

(i) $X_n \geq 0$ for all $n \in \mathbb{N}$

(ii) $(X_n)_{n\in\mathbb{N}}$ u.i.

Then: $\forall T$ stopping time (finite or not), we have $X_T \in \mathcal{L}^1$.

Moreover, $\forall S \leq T$ stopping times:

Case (i): $\mathbb{1}_{S<\infty} X_S \geq E[\mathbb{1}_{\{T<\infty\}} X_T | \mathcal{F}_S]$

Case (ii): $X_S \geq E[X_T | \mathcal{F}_S]$

Prf. Case (i). $E[X_T] \underset{\text{Fatou}}{\leq} E[\liminf_n X_{T\wedge n}] \underset{\uparrow}{\leq} E[X_0] \Rightarrow X_T \in \mathcal{L}^1$.

O.S.T for Bounded Stopping Time.

for $S \leq T$:

Case (i) for $S \le T$:

Lemma 18

- If $S \le T \le N \in \mathbb{N}$: By exercise, $\mathbb{E}[X_T \mid \mathcal{F}_S] \le X_S$.
- In general: $\forall B \in \mathcal{F}_S$: fix $\forall k \in \mathbb{N}$:

$$\mathbb{E}[X_{S\wedge k} \mathbb{1}_{B \cap \{S \le k\}}] \ge \mathbb{E}[X_{T\wedge k} \mathbb{1}_{B\cap\{T \le k\}}]$$

($B \cap \{S \le k\}$: $A \in \mathcal{F}_S$)

$$\| \qquad\qquad\qquad \|$$

$$\mathbb{E}[X_S \mathbb{1}_{B\cap\{S \le k\}}] \ge \mathbb{E}[X_T \mathbb{1}_{B\cap\{T\le k\}}]$$

$k \to +\infty$ ↓ for $X \ge 0$, Monotone c.v. ↓ ↓

$$\mathbb{E}[X_S \cdot \mathbb{1}_{B\cap\{S<+\infty\}}] \ge \mathbb{E}[X_T \mathbb{1}_{B\cap\{T<+\infty\}}]$$

$$\Rightarrow \quad X_S \cdot \mathbb{1}_{S<\infty} \ge \mathbb{E}[\mathbb{1}_{\{T<\infty\}} X_T \mid \mathcal{F}_S]$$

Case (ii): (X_n) u.i. $\Rightarrow$ bounded in $\mathcal{L}^1$ $\Rightarrow$ $(-X_n)$ bounded in $\mathcal{L}^1$.

$\Rightarrow -X_n \xrightarrow[a.s]{} -X_\infty \Rightarrow X_n \xrightarrow[a.s]{} X_\infty$. $\Longrightarrow X_n \to X_\infty$ in $\mathcal{L}^1$.
+ u.i.

Since $X_n \ge \mathbb{E}[X_{n+m} \mid \mathcal{F}_n]$ for every $m \ge 0$:

$$\mathbb{E}\big[\,|\mathbb{E}[X_{n+m} - X_\infty \mid \mathcal{F}_n]|\,\big] \le \mathbb{E}[|X_{n+m} - X_\infty|] \xrightarrow[m\to+\infty]{\mathcal{L}^1 \text{ c.v.}} 0$$

i.e. $\mathbb{E}[X_{n+m} \mid \mathcal{F}_n] \to \mathbb{E}[X_\infty \mid \mathcal{F}_n]$ in Proba.

$\Rightarrow X_n \ge \mathbb{E}[X_\infty \mid \mathcal{F}_n]$ (a subsequence $\mathbb{E}[X_{n+m_i} \mid \mathcal{F}_n] \to \mathbb{E}[X_\infty \mid \mathcal{F}_n]$ a.s.).

Let $Z_n := \mathbb{E}[X_\infty \mid \mathcal{F}_n]$. so $Z_n \to X_\infty$ a.s and in $\mathcal{L}^1$.

$Y_n := X_n - Z_n$ is a supermartingale, ≥ 0. And $Y_\infty = 0$.

By Case (i): $X_T = Y_T + Z_T \in \mathcal{L}^1$ and $Y_S \ge \mathbb{E}[Y_T \mid \mathcal{F}_S]$
By Martingale case: $Z_S = \mathbb{E}[Z_T \mid \mathcal{F}_S]$.
$\Rightarrow X_S \ge \mathbb{E}[X_T \mid \mathcal{F}_S]$ ∎

§6 Backwards Martingales

In this section, we consider time index $I = -\mathbb{N}_0 = (\cdots -n, -(n-1), \cdots, -2, -1, 0)$

Let $(\mathcal{F}_{-n}, -n \in -\mathbb{N}_0)$ be a backwards filtration: $(\cdots \subseteq \mathcal{F}_{-n} \subseteq \mathcal{F}_{-(n-1)} \cdots \subseteq \mathcal{F}_{-2} \subseteq \mathcal{F}_{-1} \subseteq \mathcal{F}_0)$

$-n \leq -m \Rightarrow \mathcal{F}_{-n} \subseteq \mathcal{F}_{-m}$. ($\mathcal{F}_{-n}$ becomes smallest when $(-n \to -\infty)$)

Let $\mathcal{F}_{-\infty} = \bigcap_{-n \in -\mathbb{N}_0} \mathcal{F}_{-n}$. (As an intersection, it is a σ-algebra).

Def 30 (Backwards Martingales)

1) Let $(\mathcal{F}_{-n}, -n \in -\mathbb{N}_0)$ be a backwards filtration. Then $(X_{-n}, -\mathbb{N}_0)$ is a backwards martingale,

if X_{-n} is $\mathcal{F}_{-n}$-measurable for each $-n \in \mathbb{N}_0$, $\mathbb{E}[|X_{-n}|] < +\infty$,

And for each $-n \leq -m$:

$$X_{-n} = \mathbb{E}[X_{-m} \mid \mathcal{F}_{-n}]$$

2) If
- $\forall -n \leq -m$, $X_{-n} \geq \mathbb{E}[X_{-m} \mid \mathcal{F}_{-n}]$: backwards supermartingale. ($\mathbb{E}[X_m \mid \mathcal{F}_n] \leq X_{-n}$)
- $\forall -n \leq -m$, $\mathbb{E}[X_m \mid X_n] \geq X_{-n}$: backwards submartingale

Thm 31 Let $(X_{-n})_{-n \in -\mathbb{N}_0}$ be a backwards supermartingale. Suppose that

[H]: $\sup_{n \in -\mathbb{N}_0} \mathbb{E}[|X_n|] < +\infty$ (i.e. the family $(X_{-n})_{-n \in -\mathbb{N}_0}$ is bounded in L^1).

Then: The family $(X_{-n}, -n \in \mathbb{N}_0)$ is U.I.. Moreover,

$$\lim_{-n \to -\infty} X_{-n} = X_{-\infty} \quad \text{a.s. and in } L^1.$$

Then limit $X_{-\infty}$ satisfies

$$\mathbb{E}[X_{-n} \mid \mathcal{F}_{-\infty}] = X_{-\infty}, \quad \text{for all } n \in -\mathbb{N}_0.$$

Rmk 1) When $(X_{-n})_{-n \in \mathbb{N}_0}$ is a backwards martingale, the condition [H] is automatically satisfied: Since $X_{-n} = \mathbb{E}[X_0 \mid \mathcal{F}_{-n}]$ for all $-n \in \mathbb{N}_0$:

Then $\mathbb{E}[|X_n|] \leq \mathbb{E}[|X_0|]$.

We also have immediately the U.I., using Cor 25.

2). $X_{-\infty}$ is $\mathcal{F}_{-m}$-measurable for every $-m$: as $X_{-\infty} = \lim_{-n \to +\infty, -n \leq -m} X_n$.

So X_∞ is $\mathcal{F}_{-\infty}$-measurable

3) Note that for a "formal" martingale, condition [H] is not enough to guarantee convergence in $\mathcal{L}^1$.

Pf • We will use the idea of the proof of Thm 15: consider numbers of upcrossings.

Fix $K \in \mathbb{N}$. Let $Y_n^{(K)} := -X_{-K+n}$ $n \in \{0, 1, \dots, K\}$ ~~$n \in \mathbb{N}_0$~~

$\mathcal{G}_n^{(K)} := \mathcal{F}_{-K+n}$ $n \in \{0, 1, \dots, K\}$

for $n > K$: let $Y_n^{(K)} = -X_0$, $\mathcal{G}_n^{(K)} = \mathcal{F}_0$

Then: $(Y_n^{(K)}, n \in \mathbb{N}_0)$ is a $(\mathcal{G}_n^{(K)})$-submartingale.

We can use Lem 17, the upcrossing inequality, to deduce that: $\forall b > a \in \mathbb{Q}$

$$(b-a)\,\mathbb{E}\left[N_K([a,b], Y^{(K)})\right] \le \mathbb{E}\left[(Y_K^{(K)} - a)^+\right] = \mathbb{E}\left[(-X_0 - a)^+\right] \le |a| + \mathbb{E}[|X_0|]$$

Let $K \uparrow +\infty$:

Then $\lim_{K \uparrow +\infty} N_K([a,b], Y^{(K)}) = N_\infty([a,b], -X)$,

where the limit is the total number of upcrossings of $-X$:

$$N_\infty([a,b], -X) = \sup\{k \in \mathbb{N} : \exists -m_1 < -n_1 < \dots < -m_k < -n_k \le 0,$$
$$-X_{m_1} \le a, -X_{n_1} \ge b, \dots, -X_{m_k} \le a, -X_{n_k} \ge b\}.$$

So we have $N_\infty([a,b], -X) < +\infty$ a.s.

By similar arguments as in the proof of Thm 15, we deduce that

$$X_{-n} \xrightarrow[-n \to -\infty]{} X_{-\infty} \quad \text{a.s.}$$

Moreover, by Fatou: $\mathbb{E}[|X_{-\infty}|] < +\infty$.

• We next prove the $(X_{-n})_{-n \in -\mathbb{N}_0}$ is U.I.:

note that for $-n \le -m$,

$$\mathbb{E}[X_{-n}] \ge \mathbb{E}\left[\mathbb{E}[X_{-m} \mid \mathcal{F}_{-n}]\right] = \mathbb{E}[X_{-m}]$$ (increasing at $-n \to -\infty$) (with a upper bounded because of [H])

By [4], $\exists\ -K<0$ small enough, s.t.

$$\mathbb{E}[X_{-n}] \leq \mathbb{E}[X_{-K}] + \frac{\varepsilon}{2}. \quad ①$$

The finite family $\{X_{-K}, \dots X_{-1}, X_0\}$ is U.I., so $\exists M>0$ large enough s.t. for all $-n \in \{-K, \dots -1, 0\}$,

$$\mathbb{E}[|X_{-n}| \cdot \mathbb{1}_{\{X_{-n}>M\}}] < \varepsilon.$$

Let us look at, for $-n<-K$:

$$\mathbb{E}[|X_{-n}| \cdot \mathbb{1}_{\{X_{-n}>M\}}] = \mathbb{E}[-X_{-n} \mathbb{1}_{\{X_{-n}<-M\}}] + \mathbb{E}[X_{-n} \cdot \mathbb{1}_{\{X_{-n}>M\}}]$$

$$= -\mathbb{E}[X_{-n} \cdot \mathbb{1}_{\{X_{-n}<-M\}}] + \mathbb{E}[X_{-n}] - \mathbb{E}[X_{-n} \cdot \mathbb{1}_{\{X_{-n}\leq M\}}]$$

Using the fact $X_{-n} \geq \mathbb{E}[X_{-K} | \mathcal{F}_{-n}]$, and ①:

$$\text{RHS} \leq -\mathbb{E}[\mathbb{E}[X_{-K} | \mathcal{F}_{-n}] \cdot \mathbb{1}_{\{X_{-n}<-M\}}] + \mathbb{E}[X_{-K}] + \frac{\varepsilon}{2} - \mathbb{E}[X_{-K} \cdot \mathbb{1}_{\{X_{-n}\leq M\}}]$$

$$= -\mathbb{E}[X_{-K} \cdot \mathbb{1}_{\{X_{-n}<-M\}}] + \mathbb{E}[X_{-K} \cdot \mathbb{1}_{\{X_{-n}>M\}}] + \frac{\varepsilon}{2}$$

$$\leq \mathbb{E}[|X_{-K}| \mathbb{1}_{\{X_{-n}>M\}}] + \frac{\varepsilon}{2} \quad ②$$

Using the U.I. of $\{X_{-K}\}$: $\forall \varepsilon>0, \exists \delta>0$, s.t.

for all A with $\mathbb{P}(A)<\delta$: $\mathbb{E}[|X_{-K}| \cdot \mathbb{1}_A] < \frac{\varepsilon}{2}$. ③

Let us make M larger, s.t. $\frac{\sup_i \mathbb{E}[|X_i|]}{M} < \delta$:

Then. $\mathbb{P}(|X_{-n}|>M) \leq \frac{1}{M}\mathbb{E}[|X_{-n}|] < \delta$.

Then by ③: $② < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

So we have U.I.

- U.I. $\Rightarrow$ $X_n \to X_{-\infty}$ in $\mathcal{L}^1$.
- Finally, for $\forall A \in \mathcal{F}_{-\infty}$: $\forall -m \leq -n$, by Backwards supermartingale,

$$\mathbb{E}[X_{-m} \cdot \mathbb{1}_A] \geq \mathbb{E}[X_{-n} \mathbb{1}_A].$$

(by $\mathcal{L}^1$ c.v.) $\downarrow m \to +\infty$

$\mathbb{E}[X_{-\infty} \cdot \mathbb{1}_A]$. This means $X_{-\infty} \geq \mathbb{E}[X_n | \mathcal{F}_{-\infty}]$

Cor 32. Let $Z \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{G}_n)_{n\in\mathbb{N}}$ is a sequence of decreasing σ-algebras. Then:

$$\mathbb{E}[Z \mid \mathcal{G}_n] \xrightarrow[n\to+\infty]{\text{a.s and in } L^1} \mathbb{E}[Z \mid \mathcal{G}_\infty],$$

where $\mathcal{G}_\infty = \bigcap_{n\in\mathbb{N}} \mathcal{G}_n$.

Pf: $X_{-n} := \mathbb{E}[Z \mid \mathcal{G}_n]$ $\mathcal{F}_{-n} := \mathcal{G}_n$. Then (X_{-n}) is a Backwards martingale.

$\Rightarrow$ $X_{-n} \to X_{-\infty}$ a.s in L^1, and $X_{-\infty} = \mathbb{E}[X_{-1} \mid \mathcal{G}_\infty]$

$= \mathbb{E}[Z \mid \mathcal{G}_\infty]$ ∎

Applications of Backwards Martingales.

1) SLLN. $(\xi_i)_{i\geq 1}$ i.i.d., $\mathbb{E}[|\xi_1|] < +\infty$. $S_n := \sum_{i=1}^n \xi_i$.

Note that, $\mathbb{E}[\xi_1 \mid S_n] = \frac{1}{n} S_n$. (Ex Sheet 2)

We next prove: for all $n \geq 1$, $\mathbb{E}[\xi_1 \mid S_n, \xi_{n+1}, \xi_{n+2}, \dots] = \frac{1}{n} S_n$. ④

Let $Z := \xi_1$, $\mathcal{H}_1 = \sigma(S_n)$ $\mathcal{H}_2 = \sigma(\xi_{n+1}, \xi_{n+2}, \dots)$.

Then ④ is a consequence of the following Lemma:

Lem 33. $Z \in \mathcal{L}^1$. $\mathcal{H}_1 \in \mathcal{F}$, $\mathcal{H}_2 \in \mathcal{F}$ are two sub σ-algebras. Suppose that $\mathcal{H}_2$ is independent of $\sigma(\sigma(Z) \cup \mathcal{H}_1)$

Then: $\mathbb{E}[Z \mid \sigma(\mathcal{H}_1 \cup \mathcal{H}_2)] = \mathbb{E}[\mathcal{H}_1]$

Pf of Lem 33. Using Monotone class Thm (π-λ Thm).

Let $\mathcal{A} = \{B \cap C : B \in \mathcal{H}_1, C \in \mathcal{H}_2\}$. Then $\mathcal{A}$ is a π-system.

$\mathcal{B} := \{A \in \sigma(\mathcal{H}_1 \cup \mathcal{H}_2) : \mathbb{E}[\mathbb{1}_A Z] = \mathbb{E}[\mathbb{1}_A \cdot \mathbb{E}[Z \mid \mathcal{H}_1]]\}$.

Then $\mathcal{B}$ is a λ-system and $\mathcal{A} \subseteq \mathcal{B}$.

$\Rightarrow$ $\sigma(\mathcal{A}) \subseteq \mathcal{B}$. ∎

We next prove the SLLN by using the Backwards martingale: $\frac{1}{n} S_n = \mathbb{E}[Z \mid \mathcal{G}_n]$, where $Z = \xi_1$, $\mathcal{G}_n = \sigma(S_n, \xi_{n+1}, \xi_{n+2}, \cdots)$.

Then: By Cor 32, $\frac{1}{n} S_n \to \mathbb{E}[Z \mid \mathcal{G}_\infty]$ a.s. and in L^1,

with $\mathcal{G}_\infty = \bigcap_{n \geq 0} \mathcal{G}_n$. But $\mathcal{G}_\infty$ is trivial by Kolmogorov 0-1 law.

So $\mathbb{E}[Z \mid \mathcal{G}_\infty] = \mathbb{E}[Z] = \mathbb{E}[\xi_1]$.

2) Hewitt-Savage 0-1 law.

Let $(\xi_i)_{i \in \mathbb{N}}$ be i.i.d. r.v.s on $(E, \mathcal{E})$. In other words, $(\xi_i)_{i \in \mathbb{N}}$ is a r.v. on $(E^{\mathbb{N}}, \mathcal{E}^{\mathbb{N}})$.

Let F be a measurable function $(E^{\mathbb{N}}, \mathcal{E}^{\mathbb{N}}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Then F is called symmetric, if

$$F(x_1, x_2, x_3 \dots) = F(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, \dots),$$

for every ~~perm~~ finite permutation of $\mathbb{N}$.

Thm 34 (Hewitt-Savage 0-1 law) Let F be a symmetric function $(E^{\mathbb{N}}, \mathcal{E}^{\mathbb{N}}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $(\xi_i)_{i \in \mathbb{N}}$ i.i.d on $(E, \mathcal{E})$. Then the random variable $F(\xi_1, \xi_2, \dots)$ is a.s. a constant.

Eg of Thm 34: If (ξ_i) i.i.d in $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and consider a RW in $\mathbb{R}^d$.

$S_n = \sum_{i=1}^{n} \xi_i$. Let ~~$B \subseteq \mathbb{R}^d$~~ $B \in \mathcal{B}(\mathbb{R}^d)$.

Then: $\mathbb{1}_{\{\#\{n \geq 1, S_n \in B\} = \infty\}}$ is symmetric. Therefore, by Thm 34, we have $\mathbb{P}(\#\{n \geq 1 : S_n \in B\} = \infty) = 0$ or 1.

Proof of Thm 34

Without loss of generality, we may assume F is bounded.

Let $\mathcal{F}_n = \sigma(\xi_1, \cdots \xi_n)$ $\mathcal{G}_n = \sigma(\xi_{n+1}, \xi_{n+2}, \cdots)$.

Denote $Y := F(\xi_1, \xi_2, \cdots)$.

Let $X_n := E[Y | \mathcal{F}_n]$ & $Z_n := E[Y | \mathcal{G}_n]$.

By Thm 27: since X_n is a closed martingale, ~~$X_n \to Y$ as~~

$$X_n \to E[Y | \mathcal{F}_\infty] = Y \text{ a.s. and in } L^1.$$

By Cor 32: The backwards martingale $Z_n \to \mathbb{E}[Y | \mathcal{G}_\infty]$ a.s. and in L^1.

$\mathbb{E}[Y] = \mathbb{E}[Y|\mathcal{G}_\infty]$ ↑ By Kolmogorov 0-1 law.

By the two L^1-convergence:

$\forall \varepsilon > 0$, $\exists N_0$ large enough, s.t. $\forall n - N_0$,

$$E[|X_n - Y|] < \varepsilon, \quad \mathbb{E}[|Z_n - \mathbb{E}[Y]|] < \varepsilon. \quad (5)$$

On the other hand, since X_n is $\mathcal{F}_n$-measurable, $\exists g: E^n \to \mathbb{R}$ measurable,

s.t. $X_n = g(\xi_1, \cdots \xi_n)$.

So (5) tells us that $\mathbb{E}[|g(\xi_1 - \xi_n) - F(\xi_1, \cdots \xi_n, \cdots)|] < \varepsilon$.

~~Since~~ But F is symmetric:

We have that $F(\xi_{n+1}, \cdots \xi_{2n}, \xi_1, \cdots \xi_n, \xi_{2n+1}, \xi_{2n+2} \cdots)$

$$= F(\xi_1 \cdots \xi_n, \xi_{n+1} \cdots \xi_{2n}, \xi_{2n+1}, \cdots)$$

$$= Y.$$

Therefore, we also have $\mathbb{E}[|g(\xi_{n+1}, \cdots \xi_{2n}) - Y|] < \varepsilon$. (6)

Taking Cond. Expection $/ \mathcal{G}_n$ to (6), + Jensen:

$$E[|\mathbb{E}[Y|\mathcal{G}_n] - E[g(\xi_{n+1} \cdots \xi_{2n}) | \mathcal{G}_n]|] < \varepsilon.$$

i.e., $E[|Z_n - g(\xi_{n+1} \cdots \xi_{2n})|] < \varepsilon$. (7)

Using (5) (6) (7): $E[|Y - E[Y]|] < 3\varepsilon$.

ε is arbitrary $\therefore$ $Y = E[Y]$ a.s. This ends the proof. ∎